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Hölder estimates for singular non-local parabolic equations

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Abstract

In this paper, we establish local Hölder estimate for non-negative solutions of the singular equation (M.P) below, for m in the range of exponents $(\frac{n-2\sigma}{n+2\sigma}, 1)$. Since we have trouble in finding the local energy inequality of v directly, we use the fact that the operator $(-\Delta)^{\sigma}$ can be thought as the normal derivative of some extension v^* of v to the upper half space (Caffarelli and Silvestre, 2007 [5]), i.e., v is regarded as boundary value of v^* the solution of some local extension problem. Therefore, the local Hölder estimate of v can be obtained by the same regularity of v^*. In addition, it enables us to describe the behavior of solution of non-local fast diffusion equation near their extinction time.

Keywords: Fractional Laplacian; Extension problem; Fully non-linear parabolic equations; Porous medium equation; Fast diffusion equation

1. Introduction

We consider initial value problem with fractional fast diffusion:

$$\begin{cases} (-\Delta)^{\sigma} u^m + u_t = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \backslash \Omega, \\ u(x,0) = u_0(x) & \text{non-negative and } \dot{H}_0^{\sigma}\text{-bounded} \end{cases} \tag{1.1}$$

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in the range of exponents $\frac{n-2\sigma}{n+2\sigma} < m < 1$, with $0 < \sigma < 1$. The fractional Laplacian of a function $f : \mathbb{R}^n \to \mathbb{R}$ is expressed by the formula

$$(-\Delta)^\sigma f(x) = C_{n,\sigma} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2\sigma}}\, dy$$

where $C_{n,\sigma}$ is some normalization constant. In addition, the norm in $\dot{H}^\sigma$ is given precisely by

$$\|f\|_{\dot{H}^\sigma} = \sqrt{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\sigma}}\, dx\, dy} \cong \sqrt{\int_{\mathbb{R}^n} |\xi|^{2\sigma} \left|\hat{f}(\xi)\right|^2 d\xi} \tag{1.2}$$

for Fourier transform of f in x. Note that the Sobolev embedding results say that $\dot{H}^\sigma \subset L^{2n/(n-2\sigma)}$ (Chap V in [16]).

The regularity of harmonic functions with respect to non-local operators was studied in several recent papers like [2] and [3], however their point of view is probabilistic. The pure analytic point of view for regularity appears in [15], which deals with non-divergence structure. Caffarelli and Vasseur [6] show that, using Digiorgi type iterative techniques, a certain class of weak solutions of the quasi-geostrophic equation with initial L^2 data and critical diffusion $(-\Delta)^{\frac{1}{2}}$ gain local Hölder regularity for any space dimension. In [6], they consider the harmonic extension problem corresponding to the original one in order to avoid difficulties stem from the non-locality. In [14], they develop a theory of existence, uniqueness and regularity for fractional porous medium equation.

We are interested in studying of the properties of non-linear eigenvalue problems with fractional powers: regularities, geometric properties, etc. However, there are lots of difficulties to be solved directly. For example, the geometric properties of the first eigenfunction for the fractional Laplacian $-(-\Delta)^{\frac{b}{2}}$ are sill unsolved (Conjecture 1.1 in [1]).

For this reason, in this work, we will deal with a Hölder regularity of $v = u^m$, which is a solution of

$$\begin{cases} (-\Delta)^\sigma v + \left(v^{\frac{1}{m}}\right)_t = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \\ v(x,0) = v_0(x) = u_0^m(x) & \text{in } \Omega, \end{cases} \tag{M.P}$$

assuming that the initial value v_0 is strictly positive in the interior of Ω in $\mathbb{R}^n$. Main two theorems state as follows:

Theorem 1.1 *(Boundedness for positive times). Let $v(x,t)$ be a function in $L^\infty(0,T; L^{\frac{2n}{n-2\sigma}}(\Omega)) \cap L^2(0,T; \dot{H}_0^\sigma(\mathbb{R}^n))$, then*

$$\sup_{x\in\Omega} \left|v(x,T)\right| \leqslant C^* \frac{\|v_0\|_{L^{\frac{2n}{n-2\sigma}}(\Omega)}}{T^{\frac{mn}{2mn-(n-2\sigma)(1+m)}}}$$

for some constant $C^* > 0$.

For the second theorem, we need better control of v.

Theorem 1.2 *(Hölder regularity of fractional FDE). For $x_0 = (x_0^1, \dots, x_0^n)$, we define $Q_r(x_0, t_0) = [x_0^i - r, x_0^i + r]^n \times [t_0 - r^{2\sigma}, t_0]$, for $t_0 > r^{2\sigma} > 0$. Assume now that $[x_0^i - r, x_0^i + r]^n \subset \Omega$ and $v(x,t)$ is bounded in $\mathbb{R}^n \times [t_0 - r^{2\sigma}, t_0]$, then there exist constants γ and β in $(0,1)$ that can be determined a priori only in terms of the data, such that v is C^β in $Q_{\gamma r}(x_0, t_0)$.*

In order to develop the Hölder regularity method, it is necessary to localize the energy inequality by space and time truncation. Due to the non-locality of the diffusion, this appears complicated. On the other hand, $(-\Delta)^\sigma v$ can be thought as the normal derivative of some extension of v (the Dirichlet to Neumann operator of v; see [5] for a general discussion). This allows us to realize the truncation as a standard local equation in one more dimension: we introduce first the corresponding extension v^* defined from $C_0^\infty(\mathbb{R}^n)$ to $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^+)$ by

$$-\nabla\big(y^a \nabla v^*\big) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty),$$
$$v^*(x, 0) = v(x) \quad \text{for } x \in \mathbb{R}^n$$

for $a = 1 - 2\sigma$. Then the following result holds true: for v defined on $\mathbb{R}^n$, we have:

$$(-\Delta)^\sigma v(x) = \partial_\nu v^*(x, 0) = -\lim_{y \to 0} y^a v_y^*(x, y)$$

where we denote by $\partial_\nu v^*$ the outward normal derivative of v^* on the boundary $\{y = 0\}$. Hence, it is possible to consider the solution v of problem (M.P) as the boundary value of v^* which is solution of

$$\begin{cases} \nabla\big(y^a \nabla v^*\big) = 0 & \text{in } y > 0, \\ \lim\limits_{y \to 0} y^a v_y^*(x, y, t) = \big(v^{\frac{1}{m}}\big)_t(x, 0, t), & x \in \Omega, \\ v^*(x, 0, t) = 0 & \text{on } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{1.3}$$

Thus, we can obtain the Hölder estimate of v immediately by showing the Hölder regularity of v^*.

Since the diffusion coefficients $D(v) = |v|^{1-\frac{1}{m}}$ go to infinity as $v \to 0$, we need to control the oscillation of v from below. Hence, we consider the new function w^* derived from v^* such that $w^*(x, y, t) = M - v^*(x, y, t + t_0)$ with $M = M(t_0) = \sup_{t \geqslant t_0 > 0} v^*$. By Theorem 1.1, we know that the solution satisfies

$$v^*(\cdot, t) \leqslant M(t_0) < \infty \quad (t \geqslant t_0).$$

From this, we get to a familiar situation:

$$\begin{cases} \nabla\big(y^a \nabla w^*\big) = 0 & \text{in } y > 0, \\ -\lim\limits_{y \to 0^+} y^a \nabla_y w^*(x, y) = \big[\big(M - w^*\big)^{\frac{1}{m}}\big]_t(x, 0), & x \in \Omega, \\ w^*(x, 0, t) = M & \text{on } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{1.4}$$

The paper is divided into four parts: In Section 2 we study several properties for the fractional fast diffusion equation (shortly, FDE)

$$(-\Delta)^\sigma u^m + u_t = 0 \quad \left(\frac{n-2\sigma}{n+2\sigma} < m < 1\right). \tag{1.5}$$

In Section 3, we show the existence of weak solution of the problem (M.P). Also, we investigate the boundedness of the solutions of problem (M.P) for positive times. Lastly in this section, we compute local energy inequality of $(w^* - k)_\pm$ which will be a key step in establishing local Hölder estimates. The proof of the Hölder regularity of problems is given in Section 4. In this section, we consider the extension v^* of v that solves (M.P). This allows us to treat non-linear problems, involving fractional Laplacians, as a local problems. In the last section, we study the existence of non-linear eigenvalue problem with fractional powers which is asymptotic profile of the parabolic flow (M.P) on extinction time.

Notations. Before we explain the main ideas of the paper, let us summarize the notations and definitions that will be used.

- Numbers: $\frac{n-2\sigma}{n+2\sigma} < m < 1$, $\alpha = 1 - \frac{1}{m}$, $0 < \sigma < 1$, $a = 1 - 2\sigma$ and $M = \sup_{t \geqslant t_0} v^*$ for some $t_0 > 0$.
- For $x_0 = (x_0^1, \dots, x_0^n) \in \mathbb{R}^n$, we denote by $B_r(x_0) = [x_0^i - r, x_0^i + r]^n$ a cube in the x variable only, and $B_r^*(x_0) = B_r(x_0) \times (0, r) \in \mathbb{R}^n \times (0, \infty)$ a cube in the x, y variables sitting on the plane $y = 0$.
- We construct the cylinders $Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^{2\sigma}, t_0)$ and, for $\alpha = 1 - \frac{1}{m}$, $Q_r(\omega)(x_0, t_0) = B_r(x_0) \times (t_0 - \frac{r^{2\sigma}}{\omega^\alpha}, t_0)$. In addition, for ρ_1 and ρ_2, $Q_r(\omega, \rho_1, \rho_2)(x_0, t_0) = B_{r-\rho_1 r}(x_0) \times (t_0 - (1 - \rho_2)\frac{r^{2\sigma}}{\omega^\alpha}, t_0)$.
- The cylinders $Q_r^*(x_0, t_0)$, $Q_r^*(\omega)(x_0, t_0)$ and $Q_r^*(\omega, \rho_1, \rho_2)(x_0, t_0)$ are obtained in a similar manner by replacing $B_r(x_0)$ and $B_{r-\rho_1 r}(x_0)$ by $B_r^*(x_0)$ and $B_{r-\rho_1 r}^*(x_0)$ respectively.
- We let Q_r, Q_r^*, $Q_r(\omega)$, $Q_r^*(\omega)$, $Q_r(\omega, \rho_1, \rho_2)$ and $Q_r^*(\omega, \rho_1, \rho_2)$ be the cylinders around the point $(x_0, t_0) = (0, 0)$.

Let us start with showing the properties of the solution for FDE with fractional powers in the following section.

2. Properties of fast diffusion equations with fractional powers

Since the operator $(-\Delta)^\sigma$ converges to $(-\Delta)$ as $\sigma \to 1$, it is natural to expect that the solutions of (1.5) have a lot in common with those of the FDE. Hence, we will first discuss such properties for fractional FDE.

2.1. Scale invariance

Let us examine the application of *scaling transformations* to the fractional FDE in some detail. Let $u = u(x, t)$ be a solution of the fractional FDE,

$$(-\Delta)^\sigma u^m + u_t = 0 \quad (0 < m < 1). \tag{2.1}$$

We apply the group of dilations in all the variables

$$u'(x',t') = Ku(x,t), \qquad x' = Lx, \qquad t' = Tt,$$

and impose the condition that u' has to be again a solution of (2.1). Then:

$$\frac{\partial u'}{\partial t'} = \frac{K}{T}\frac{\partial u}{\partial t}\left(\frac{x'}{L}, \frac{t'}{T}\right) \tag{2.2}$$

and

$$(-\Delta)^\sigma\big[u'(x',t')\big]^m = \frac{K^m}{L^{2\sigma}}\int \frac{[u(\frac{x'}{L},\frac{t'}{T})]^m - [u(\frac{y'}{L},\frac{t'}{T})]^m}{|\frac{x'}{L}-\frac{y'}{L}|^{n+2\sigma}}\,d\left(\frac{y'}{L}\right) = \frac{K^m}{L^{2\sigma}}(-\Delta)^\sigma u^m.$$

Hence, (2.2) will be a solution if and only if $K^{m-1} = L^{2\sigma}T^{-1}$. We thus obtain a two-parametric transformation group acting on the set of solution of (2.1). Assuming that $m \neq 1$, we may choose as free parameters L and T, so that it can be written as

$$u'(x',t') = L^{\frac{2\sigma}{m-1}}T^{-\frac{1}{m-1}}u\left(\frac{x'}{L}, \frac{t'}{T}\right).$$

Using standard letters for the independent variables as putting $u' = \tau u$, we get:

$$(\tau u)(x,t) = L^{\frac{2\sigma}{m-1}}T^{-\frac{1}{m-1}}u\left(\frac{x}{L}, \frac{t}{T}\right). \tag{2.3}$$

The conclusion is:

Lemma 2.1. *If u is a solution of the fractional FDE in a certain class of solutions $\mathbb{S}$ that is closed under dilations in x, t and u, then τu given by* (2.3) *is again a solution of the fractional FDE in the same class.*

2.2. L^1-contraction

This is a very important estimate which has played a key role in the fractional powers of the FDE theory.

Lemma 2.2 *(L^1-contraction). Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary, and let u and $\tilde{u}$ be two smooth solutions of the fractional FDE*:

$$\begin{cases} (-\Delta)^\sigma u^m + u_t = 0 & \text{in } Q_T = \Omega \times (0,T), \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega \end{cases}$$

with initial date u_0, $\tilde{u}_0$ respectively. We have for every $t > \tau \geqslant 0$

$$\int_\Omega \big[u(x,t) - \tilde{u}(x,t)\big]_+\,dx \leqslant \int_\Omega \big[u(x,\tau) - \tilde{u}(x,\tau)\big]_+\,dx. \tag{2.4}$$

As a consequence,

$$\left\|u(t)-\tilde{u}(t)\right\|_1 \leqslant \|u_0-\tilde{u}_0\|_1. \tag{2.5}$$

Proof. Let $p \in C^1(\mathbb{R})$ be such that $0 \leqslant p \leqslant 1$, $p(s)=0$ for $s \leqslant 0$, $p'(s)>0$ for $s>0$. Let $w=u^m-\tilde{u}^m$ which vanishes on the boundary $\mathbb{R}^n \backslash \Omega \times [0,T)$. Subtracting the equations satisfied by u and $\tilde{u}$, multiplying by $p(w)$ and integrating in $\mathbb{R}^n$, we have for

$$\begin{aligned}
\int_{\mathbb{R}^n} (u-\tilde{u})_t\, p(w)\, dx &= \int_{\mathbb{R}^n} -(-\Delta)^\sigma w p(w)\, dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[-w(x,t)+w(y,t)]p(w(x,t))}{|x-y|^{n+2\sigma}}\, dy\, dx = A.
\end{aligned}$$

Since

$$\begin{aligned}
A &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[-w(x,t)+w(y,t)][p(w(x,t))-p(w(y,t))]}{|x-y|^{n+2\sigma}}\, dy\, dx \\
&\quad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[-w(x,t)+w(y,t)]p(w(y,t))}{|x-y|^{n+2\sigma}}\, dy\, dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[-w(x,t)+w(y,t)][p(w(x,t))-p(w(y,t))]}{|x-y|^{n+2\sigma}}\, dy\, dx - A,
\end{aligned}$$

we obtain

$$\int_{\mathbb{R}^n} (u-\tilde{u})_t\, p(w)\, dx = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[-w(x,t)+w(y,t)][p(w(x,t))-p(w(y,t))]}{|x-y|^{n+2\sigma}}\, dy\, dx.$$

Note that the term in the right-hand side is non-positive. Therefore letting p converge to the sign function sign_0^+, [17], and observing that

$$\frac{\partial}{\partial t}(u-\tilde{u})_+ = (u-\tilde{u})_t \operatorname{sign}_0^+,$$

we get

$$\frac{d}{dt}\int_{\Omega} (u-\tilde{u})_+\, dx = \frac{d}{dt}\int_{\mathbb{R}^n} (u-\tilde{u})_+\, dx \leqslant 0,$$

which implies (2.4) for $u, \tilde{u}$. To obtain (2.5), combine (2.4) applied first to u and $\tilde{u}$ and then to $\tilde{u}$ and u. $\square$

2.3. Extinction in finite time

The main difference with porous medium equation is the finite time convergence of the solutions to the zero solution. This is called *extinction in finite time* and read as follows.

Lemma 2.3. *If $u(x,t)$ is the $C^{2,1}$ solution of the fractional FDE*:

$$\begin{cases} (-\Delta)^\sigma u^m + u_t = 0 & \text{in } Q_\infty = \Omega \times (0,\infty), \\ u = 0 & \text{on } \mathbb{R}^n \backslash \Omega, \\ u(x,0) = u_0(x) \in C^0(\Omega) \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary, then there exists $T^* > 0$ such that $u(\cdot,t) = 0$ for all $t \geqslant T^*$, i.e., $\lim_{t \to T^*} \|u(\cdot,t)\|_\infty = 0$ for some $T^* > 0$. The solution can be continued past the extinction time T^* in a weak sense as $u \equiv 0$.

Proof. It is enough to construct a super-solution V with the property of extinction. We will choose the function V in the form $V(x,t) = X(x)T(t)$. Let $R > 0$ be such that $\Omega \subset B_R(0)$, and let us define functions $X(x)$ and $T(t)$ by

$$X(x) = \begin{cases} \frac{1}{R^{\frac{n-2\sigma}{m}}}, & |x| \leqslant R, \\ \frac{1}{|x|^{\frac{n-2\sigma}{m}}}, & |x| > R, \end{cases} \qquad T(t) = \begin{cases} C(T^* - t)^{\frac{1}{1-m}}, & t \leqslant T^*, \\ 0, & t > T^* \end{cases}$$

for a constant $C > 0$ we can choose later. Since

$$\begin{aligned} (-\Delta)^\sigma X^m &\geqslant \int\limits_{\mathbb{R}^n \backslash B_{2R}(0)} \frac{X^m(x) - X^m(y)}{|x-y|^{n+2\sigma}}\, dy \\ &\geqslant \left(\frac{2}{3}\right)^{n+2\sigma} \left(1 - \frac{1}{2^{n-2\sigma}}\right) \frac{1}{R^{n-2\sigma}} \int\limits_{\mathbb{R}^n \backslash B_{2R}(0)} \frac{1}{|y|^{n+2\sigma}}\, dy \\ &\geqslant \frac{1}{2\sigma} \left(\frac{2}{3}\right)^{n+2\sigma} \left(\frac{1}{2^\sigma} - \frac{1}{2^n}\right) \frac{1}{R^n} \end{aligned}$$

for all $x \in \Omega$, we have

$$(-\Delta)^\sigma V^m + V_t \geqslant C^m \bigl(T^* - t\bigr)^{\frac{m}{1-m}} \left[\frac{1}{2\sigma} \left(\frac{2}{3}\right)^{n+2\sigma} \left(\frac{1}{2^\sigma} - \frac{1}{2^n}\right) \frac{1}{R^n} - \frac{C^{1-m}}{(1-m) R^{\frac{n-2\sigma}{m}}} \right] = 0$$

if we choose $C = [(\frac{1-m}{2\sigma})(\frac{2}{3})^{n+2\sigma}(\frac{1}{2^\sigma} - \frac{1}{2^n}) R^{\frac{(1-m)n-2\sigma}{m}}]^{\frac{1}{1-m}}$. $\square$

Next we deal with the two lemmas. The ideas are based on the proof of Lemmas 1 and 2 in [4] respectively.

Lemma 2.4 *(Estimates on finite extinction time). When $\frac{n-2\sigma}{n+2\sigma} < m < 1$, there exists a positive constant C such that the solution $v = u^m$ of* (M.P) *satisfies*

$$T^* - t \leqslant C\left(\int_{\mathbb{R}^n} v^{\frac{m+1}{m}}(x,t)\,dx\right)^{\frac{1-m}{1+m}}.$$

Proof. Multiplying Eq. (M.P) by v and integrating by parts, we obtain the inequality

$$\frac{d}{dt}\left(\int_{\mathbb{R}^n} v^{\frac{m+1}{m}}\,dx\right) = -(m+1)\int_{\mathbb{R}^n} v(-\Delta)^\sigma v\,dx = -(m+1)\|v\|^2_{\dot{H}^\sigma}. \tag{2.6}$$

By the compactness of imbedding, there are constants C and C' such that, for any $v \in \dot{H}^\sigma_0$,

$$\|v\|^2_{\dot{H}^\sigma} \geqslant C\|v\|^2_{\frac{2n}{n-2\sigma}} \geqslant C'\|v\|^2_{\frac{m+1}{m}} \tag{2.7}$$

when $\frac{n-2\sigma}{n+2\sigma} < m < 1$. The last inequality makes use of Hölder inequality. Substituting (2.7) into (2.6) one obtains an inequality which can be integrated to yield

$$\left(\int_{\mathbb{R}^n} v^{\frac{m+1}{m}}\big(x,t'\big)\,dx\right)^{\frac{1-m}{1+m}} - \left(\int_{\mathbb{R}^n} v^{\frac{m+1}{m}}(x,t)\,dx\right)^{\frac{1-m}{1+m}} \leqslant -C'(1-m)\big(t'-t\big)$$

for $T^* > t' \geqslant t$. Letting $t' \to T^*$ and multiplying by -1, we have the statement of the lemma. □

Lemma 2.5. *When $\frac{n-2\sigma}{n+2\sigma} < m < 1$, the solution v of* (M.P) *satisfies*

$$\int_{\mathbb{R}^n} v^{\frac{m+1}{m}}(x,t)\,dx \leqslant \left(1 - \frac{t}{T^*}\right)^{\frac{1+m}{1-m}} \int_{\mathbb{R}^n} v^{\frac{m+1}{m}}(x,0)\,dx. \tag{2.8}$$

Proof. The proof is very similar to that of Lemma 2 in [4]. □

3. Weak solutions and local energy inequality

First, we study the problem (M.P) in the class of non-negative weak solutions. For the remainder of this paper we assume that $\frac{n-2\sigma}{n+2\sigma} < m < 1$ holds.

3.1. Weak solutions and existence

Definition 3.1. A non-negative weak solution of Eq. (M.P) is a locally integrable function, $v \in L^1_{loc}(\mathbb{R}^n \times [0,\infty))$, such that $(-\Delta)^\sigma v \in L^1_{loc}(\mathbb{R}^n \times [0,\infty))$ and $v = 0$ on $\mathbb{R}^n \setminus \Omega$, and the identity

$$\int_0^\infty \int_\Omega v^{\frac{1}{m}} \eta_t \, dx\, dt = \int_0^\infty \int_\Omega v\big[-(-\Delta)^\sigma \eta\big]\, dx\, dt \tag{3.1}$$

holds for any test function $\eta \in C_c^{2,1}(\Omega \times [0,\infty))$ with $\eta = 0$ on $\mathbb{R}^n \setminus \Omega$.

We show the existence and comparison result for weak solutions. The proof is similar to the proof of Theorem 5.5 in [17].

Lemma 3.2. *There exists a non-negative weak solution of* (M.P). *Moreover, the comparison principle holds for these solutions*: *v, $\hat{v}$ are weak solutions with initial data such that $v_0 \leqslant \hat{v}_0$ a.e. in Ω, then $v \leqslant \hat{v}$ a.e. for all $t > 0$.*

Proof. By Theorem 1.2 in [11], we can obtain

$$\int_0^\infty \int_\Omega \big(v_n^{\frac{1}{m}}\big)\eta_t \, dx\, dt = \int_0^\infty \int_\Omega v_n\big[-(-\Delta)^\sigma \eta\big]\, dx\, dt$$

for a test function η in $\Omega \times [0,\infty)$. Hence, following the proof of Theorem 5.5 in [17], we can get a desired result. □

3.2. L^∞ bounds

This subsection is devoted to the proof of Theorem 1.1. Following the similar arguments as in Section 2 in [6], we can get a desired result.

Proof of Theorem 1.1. We use the energy inequality for the levels

$$C_k = N\left(1 - \frac{1}{2^k}\right)$$

where N will be chosen later. Multiplying the equation in (M.P) by the function $v_k = (v - C_k)_+$ and integrating in space, $\mathbb{R}^n$, we have

$$\frac{1}{m}\int_{\mathbb{R}^n} \frac{d}{dt}\left[\int_0^{v_k} (\xi + C_k)^{\frac{1}{m}-1}\xi \, d\xi\right] dx + \int_{\mathbb{R}^n} v_k\big[(-\Delta)^\sigma v_k\big]\, dx \leqslant 0 \tag{3.2}$$

since

$$(C_k + v_k)^{\frac{1}{m}} v_k = (C_k + \xi)^{\frac{1}{m}} \xi \Big|_{\xi=0}^{\xi=v_k} = \int_0^{v_k} \frac{d}{d\xi}\big[(C_k+\xi)^{\frac{1}{m}}\xi\big]\, d\xi.$$

Let us fix a $t_0 > 0$, we want to show that v is bounded for $t > t_0$. For

$$T_k = t_0\left(1 - \frac{1}{2^k}\right),$$

we integrate (3.2) in time between s, $T_{k-1} < s < T_k$, and $t > T_k$ and between s and $+\infty$. Then we find

$$\sup_{t \geqslant T_k} \int_{\mathbb{R}^n} \left[\int_0^{v_k} (\xi + C_k)^{\frac{1}{m}-1} \xi \, d\xi \right] dx + m \int_{T_k}^{\infty} \|v_k\|_{\dot{H}^\sigma}^2 \, dt \leqslant 2 \int_{\mathbb{R}^n} \left[\int_0^{v_k} (\xi + C_k)^{\frac{1}{m}-1} \xi \, d\xi \right] (s) \, dx.$$

This leads to

$$\sup_{t \geqslant T_k} \int_{\Omega} v_k^{\frac{1}{m}+1} \, dx + (m+1) \int_{T_k}^{\infty} \|v_k\|_{\dot{H}^\sigma}^2 \, dt \leqslant (m+1) \int_{\Omega} v^{\frac{1}{m}-1}(s) v_k^2(s) \, dx$$

since v_k has compact support in Ω. In addition, Hölder's inequality gives

$$\begin{aligned} \int_{\Omega} v^{\frac{1}{m}-1}(s) v_k^2(s) \, dx &\leqslant \left(\int_{\Omega} v^{\frac{1-m}{m} \cdot \frac{1+m}{1-m}}(s) \, dx \right)^{\frac{1-m}{1+m}} \left(\int_{\Omega} v_k^{2 \cdot \frac{1+m}{2m}}(s) \, dx \right)^{\frac{2m}{1+m}} \\ &\leqslant C \left(\int_{\Omega} v_k^{\frac{1+m}{m}}(s) \, dx \right)^{\frac{2m}{1+m}} \end{aligned}$$

for some constant $C > 0$. Hence, for the level set of energy

$$U_k = \sup_{t \geqslant T_k} \int_{\Omega} v_k^{\frac{1}{m}+1} \, dx + (m+1) \int_{T_k}^{\infty} \|v_k\|_{\dot{H}^\sigma}^2 \, dt,$$

we have

$$U_k \leqslant 2C(m+1) \left(\int_{\Omega} v_k^{\frac{1+m}{m}}(s) \, dx \right)^{\frac{2m}{1+m}}.$$

Taking the mean value in s on $[T_{k-1}, T_k]$, we find

$$U_k \leqslant \frac{C 2^{k+1}(m+1)}{t_0} \int_{T_{k-1}}^{T_k} \left(\int_{\Omega} v_k^{\frac{1+m}{m}} \, dx \right)^{\frac{2m}{1+m}} dt \leqslant C' \left(\frac{2^k}{t_0} \right)^{1 - \frac{1-m}{1+m}} \left(\int_{T_{k-1}}^{\infty} \int_{\Omega} v_k^{\frac{1+m}{m}} \, dx \, dt \right)^{\frac{2m}{1+m}}$$

for some constant $C' = C'(\Omega, m, n, \sigma)$. We want to control the right-hand side by U_{k-1}. Sobolev and interpolation inequalities give (see interpolation inequalities of L^p-space in Lemma 3.3):

$$U_{k-1} \geqslant C \|v_{k-1}\|_{L^{2(1+\frac{(1+m)\sigma}{mn})}([T_{k-1}, \infty) \times \mathbb{R}^n)}^2.$$

Note that if $v_k > 0$, then $v_{k-1} \geqslant \frac{N}{2^k}$. Thus

$$\mathbf{1}_{\{v_k>0\}} \leqslant \left(\frac{2^k}{N} v_{k-1}\right)^{2(1+\frac{(1+m)\sigma}{mn})-\frac{1+m}{m}}.$$

Hence

$$\begin{aligned} U_k &\leqslant C'\left(\frac{2^k}{t_0}\right)^{1-\frac{1-m}{1+m}} \left(\int_{T_{k-1}}^{\infty}\int_{\Omega} v_{k-1}^{\frac{1+m}{m}} \cdot \mathbf{1}_{\{v_k>0\}}\, dx\, dt\right)^{\frac{2m}{1+m}} \\ &\leqslant C' \frac{(2^{\frac{4m}{1+m}(1+\frac{(1+m)\sigma}{mn})-\frac{2}{1+m}})^k}{t_0^{1-\frac{1-m}{1+m}} N^{\frac{4m}{1+m}(1+\frac{(1+m)\sigma}{mn})-2}} \left(\int_{\Omega} v_{k-1}^{2(1+\frac{(1+m)\sigma}{mn})}\, dx\, dt\right)^{\frac{2m}{1+m}} \\ &\leqslant C' \frac{(2^{\frac{4m}{1+m}(1+\frac{(1+m)\sigma}{mn})-\frac{2}{1+m}})^k}{t_0^{1-\frac{1-m}{1+m}} N^{\frac{4m}{1+m}(1+\frac{(1+m)\sigma}{mn})-2}} U_{k-1}^{\frac{2m}{1+m}(1+\frac{(1+m)\sigma}{mn})}. \end{aligned}$$

Note that $\frac{2m}{1+m}(1+\frac{(1+m)\sigma}{mn}) > 1$. Thus, for N such that $N^2 t_0^{\frac{2mn}{2mn-(n-2\sigma)(1+m)}}$ big enough (depending on U_0), Lemma 4.1 of Chapter I in [9], we have U_k which converges to zero. This gives $v \leqslant N$ for $t \geqslant t_0$. Hence we come to the conclusion using the fact that $U_0 \leqslant \|v(\cdot,0)\|^2_{L^{\frac{2n}{n-2\sigma}}}$. $\square$

3.3. *Local Energy Estimate of w^**

The rest of this section is devoted to the proof of the Sobolev and local energy inequalities for the extension $w^*(x,y,t) = M - v^*(x,y,t+t_0)$ with $M = \sup_{t\geqslant t_0>0} v^*$. The effect of the non-local part of $(-\Delta)^\sigma$ becomes encoded locally in the extra variable. The first result, Sobolev inequality, states as follows:

Lemma 3.3 *(Sobolev inequality). For a cut-off function η compactly supported in B_r,*

$$\|\eta v\|_{L^{\frac{2n}{n-2\sigma}}(\mathbb{R}^n)} \leqslant C\|\eta v\|_{\dot{H}^\sigma(\mathbb{R}^n)} \tag{3.3}$$

and

$$\begin{aligned} &\|\eta v\|^2_{L^2(t_1,t_2;L^2(\mathbb{R}^n))} \\ &\quad \leqslant C\Big(\sup_{t_1\leqslant t\leqslant t_2} \|\eta v\|^2_{L^2(\mathbb{R}^n)} + \big\|\nabla(\eta v)^*\big\|^2_{L^2(t_1,t_2;L^2(B_r^*,y^a))}\Big)\big|\{\eta v>0\}\big|^{\frac{2\sigma}{n+2\sigma}} \end{aligned} \tag{3.4}$$

for some $C > 0$.

Proof. The first inequality is a well-known result. The Sobolev embedding results say that $\dot{H}^\sigma \subset L^{2n/(n-2\sigma)}$ (see, for example, Chapter V in [16]).

To prove the second inequality, let $\chi_{\eta v}(x,t)$ be the function with

$$\chi_{\eta v} = \begin{cases} 1, & \eta v > 0, \\ 0, & \eta v = 0, \end{cases}$$

then we have

$$\|\eta v\|^2_{L(t_1,t_2;L^2(\mathbb{R}^n))} = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} |\eta v|^2\,dx\,dt = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} |\eta v|^2 \chi_{\eta v}\,dx\,dt.$$

Thus, by the Hölder inequality, we obtain

$$\begin{aligned} \|\eta v\|^2_{L(t_1,t_2;L^2(\mathbb{R}^n))} &\leqslant \int_{t_1}^{t_2}\left(\int_{\mathbb{R}^n} |\eta v|^{2\cdot(\frac{n+2\sigma}{n})}\,dx\right)^{\frac{n}{n+2\sigma}}\left(\int_{\mathbb{R}^n} (\chi_{\eta v})^{\frac{n+2\sigma}{2\sigma}}\,dx\right)^{\frac{2\sigma}{n+2\sigma}} dt \\ &\leqslant \left(\int_{t_1}^{t_2}\int_{\mathbb{R}^n} |\eta v|^{2\cdot(\frac{n+2\sigma}{n})}\,dx\,dt\right)^{\frac{n}{n+2\sigma}} \left|\{\eta v > 0\}\right|^{\frac{2\sigma}{n+2\sigma}}. \end{aligned}$$

Now we use interpolation inequalities of L^p spaces,

$$\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^n} |\eta v|^{2\cdot(\frac{n+2\sigma}{n})}\,dx\,dt\right)^{\frac{n}{n+2\sigma}} \leqslant \left[\int_{t_1}^{t_2}\left(\int_{\mathbb{R}^n} |\eta v|^2\,dx\right)^{(1-\beta)p}\left(\int_{\mathbb{R}^n} |\eta v|^{2q}\,dx\right)^{\frac{\beta p}{q}} dt\right]^{\frac{1}{p}}$$

where $1 < p = \frac{n+2\sigma}{n} < q$ and $\frac{1}{p} = \frac{\beta}{q} + \frac{1-\beta}{1}$ ($\beta = \frac{1}{p}$). Thus

$$\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^n} |\eta v|^{2\cdot(\frac{n+2\sigma}{n})}\,dx\,dt\right)^{\frac{n}{n+2\sigma}} \leqslant \sup_{t_1\leqslant t\leqslant t_2}\left(\int_{\mathbb{R}^n} |\eta v|^2\,dx\right) + \int_{t_1}^{t_2}\left(\int_{\mathbb{R}^n} |\eta v|^{2q}\,dx\right)^{\frac{1}{q}} dt,$$

where $q = \frac{n}{n-2\sigma}$. From the first Sobolev inequality (3.3), we have

$$\left(\int_{\mathbb{R}^n} |\eta v|^{2q}\,dx\right)^{\frac{1}{q}} \leqslant C\|\eta v\|^2_{\dot{H}^\sigma} = C\int_{\mathbb{R}^n} \eta v(-\Delta)^\sigma(\eta v)\,dx = C\int_0^\infty\int_{\mathbb{R}^n} y^a\left|\nabla(\eta v)^*\right|^2\,dx\,dy,$$

which gives the desired result. □

Next, we will derive Local Energy Estimate in the interior of $\Omega \times (0,\infty)$ which will be the main tools in establishing local Hölder estimates for the solutions. Assume that the cube $B_r = [-r,r]^n \subset \Omega$.

Lemma 3.4 *(Local Energy Estimate). Let t_1, t_2 be such that $t_1 < t_2$ and let $v^* \in L^\infty(t_1, t_2; L^2(\mathbb{R}^n \times \mathbb{R}^+))$ be solution to (1.3) and let $w^*(x, y, t) = M - v^*(x, y, t + t_0)$ with $M = \sup_{t \geqslant t_0 > 0} v^*$. Then, there exists a constant λ such that for every $t_1 \leqslant t \leqslant t_2$ and cut-off function η such that the restriction of $\eta(w^* - k)_\pm$ on B_r^* is compactly supported in $B_r \times (-r, r)$:*

$$\begin{aligned}
&\frac{1}{m} \int\limits_{B_r \times \{t_2\}} \eta^2 \left[\int\limits_0^{(w-k)_\pm} (M - k \mp \xi)^{\frac{1}{m}-1} \xi \, d\xi \right] dx + \int\limits_{t_1}^{t_2} \int\limits_{B_r^*} \left|\nabla\big(\eta\big(w^* - k\big)_\pm\big)\right|^2 y^a \, dx\, dy\, dt \\
&\quad \leqslant \int\limits_{t_1}^{t_2} \int\limits_{B_r^*} \left|(\nabla\eta)\big(w^* - k\big)_\pm\right|^2 y^a \, dx\, dy\, dt \\
&\qquad + \frac{2}{m} \int\limits_{t_1}^{t_2} \int\limits_{B_r} \left[\int\limits_0^{(w-k)_\pm} (M - k \mp \xi)^{\frac{1}{m}-1} \xi \, d\xi \right] |\eta\eta_t| \, dx\, dt \\
&\qquad + \frac{1}{m} \int\limits_{B_r \times \{t_1\}} \eta^2 \left[\int\limits_0^{(w-k)_\pm} (M - k \mp \xi)^{\frac{1}{m}-1} \xi \, d\xi \right] dx. \qquad (3.5)
\end{aligned}$$

Proof. We have for every $t_1 < t < t_2$:

$$\begin{aligned}
0 &= \int\limits_{B_r^*} \eta^2 \big(w^* - k\big)_+ \nabla\big(y^a \nabla w^*\big) \, dx\, dy \\
&= -\int\limits_{B_r^*} y^a \left|\nabla\big(\eta\big(w^* - k\big)_+\big)\right|^2 dx\, dy + \int\limits_{B_r^*} y^a \big(\big(w^* - k\big)_+\big)^2 |\nabla\eta|^2 \, dx\, dy \\
&\quad + \int\limits_{B_r} \eta^2 (w - k)_+ (-\Delta)^\sigma w \, dx.
\end{aligned}$$

Using Eq. (1.4), we find that

$$\begin{aligned}
\int\limits_{B_r} \eta^2 (w - k)_+ (-\Delta)^\sigma w \, dx &= \int\limits_{B_r} \eta^2 (w - k)_+ \big[(M - w)^{\frac{1}{m}}\big]_t \, dx \\
&= \int\limits_{B_r} \eta^2 (w - k)_+ \big[\big(M - k - (w - k)_+\big)^{\frac{1}{m}}\big]_t \, dx \\
&= \int\limits_{B_r} \eta^2 \bigg[(w - k)_+ \big(M - k - (w - k)_+\big)^{\frac{1}{m}} \\
&\qquad + \frac{m}{m+1} \big(M - k - (w - k)_+\big)^{\frac{m+1}{m}} - \frac{m}{m+1} (M - k)^{\frac{m+1}{m}} \bigg]_t \, dx.
\end{aligned}$$

Since

$$\begin{aligned}\big(M-k-(w-k)_+\big)^{\frac{1}{m}}(w-k)_+ &= (M-k-\xi)^{\frac{1}{m}}\xi\big|_{\xi=0}^{\xi=(w-k)_+}\\ &= \int_0^{(w-k)_+}\frac{d}{d\xi}\big[(M-k-\xi)^{\frac{1}{m}}\xi\big]\,d\xi,\end{aligned}$$

we get the desired result from

$$\begin{aligned}&(w-k)_+\big(M-k-(w-k)_+\big)^{\frac{1}{m}}+\frac{m}{m+1}\big(M-k-(w-k)_+\big)^{\frac{m+1}{m}}-\frac{m}{m+1}(M-k)^{\frac{m+1}{m}}\\ &\quad=-\frac{1}{m}\int_0^{(w-k)_+}(M-k-\xi)^{\frac{1}{m}-1}\xi\,d\xi.\end{aligned}$$

The estimate of $(w-k)_-$ can be obtained in a similar manner. □

4. Hölder regularity of w^*

From now on, we begin the story of Hölder continuity of w^* the solution of (1.4). In order to develop the Hölder regularity method, we get over the two humps: non-local diffusion and degeneracy. We first follow Caffarelli and Vasseur's ideas [6] to solve a difficulty stemming from non-linear evolution equations with fractional diffusion. Also we use the technique developed in [9,10,7] to overcome the degeneracy of equation.

The key idea of the proof is to work with cylinders whose dimensions are suitably rescaled to reflect the degeneracy exhibited by the equation. To make this precise, fix $(x_0,t_0)\in\Omega\times(0,T]$, for some $T>0$, and construct the cylinder

$$Q_{2R}(x_0,t_0)\subset\Omega\times(0,T].$$

After a translation we may assume that $(x_0,t_0)=(0,0)$. Set

$$\mu^+=\sup_{Q_{2R}^*}w^*,\qquad \mu^-=\inf_{Q_{2R}^*}w^*,\qquad \omega=\operatorname*{osc}_{Q_{2R}^*}w^*=\mu^+-\mu^-$$

and construct the cylinder

$$Q_R^*(\theta_0)=B_R^*\times\left(-\frac{R^{2\sigma}}{\theta_0^\alpha},0\right),\quad \theta_0=\frac{\omega}{A},\ \alpha=1-\frac{1}{m}$$

where A is a constant to be determined later only in terms of the data. We will assume that

$$\left(\frac{\omega}{A}\right)^\alpha\geqslant 1.$$

This implies

$$Q_R^*(\theta_0) \subset Q_{2R}^* \quad \text{and} \quad \operatorname*{osc}_{Q_R^*(\theta_0)} w^* \leqslant \omega.$$

Then, the first alternative in this section states as follows:

Lemma 4.1. *There exist positive numbers ρ and λ independent of $\mu^{\pm}$ and ω such that if*

$$\left|\left\{(x,t) \in Q_R(\theta_0);\ w(x,t) > \mu^+ - \frac{\omega}{2}\right\}\right| < \rho\left|Q_R(\theta_0)\right| \tag{4.1}$$

then

$$w(x,t) < \mu^+ - \frac{\lambda\omega}{4}, \quad \forall (x,t) \in Q_{\frac{R}{2}}(\theta_0).$$

Proof. Step 1. Useful barrier functions: Consider the function $h_1(x,y)$, defined by

$$\begin{aligned} \nabla\left(y^a \nabla h_1\right) &= 0 \quad \text{in } B_1^*, \\ h_1 &= 1 \quad \text{on } \partial B_1^* \cap \{y > 0\}, \\ h_1 &= 0 \quad \text{on } y = 0. \end{aligned} \tag{4.2}$$

Then, following directly the maximum principle, there is uniform constant $0 < \lambda < 1/2$ such that

$$h_1(x,y) \leqslant (1 - 2\lambda) \quad \text{on } B_{1/2}^*.$$

For the scaling invariance of the equation for h_1,

$$h_{1,R}(x,y) = h_1\left(\frac{x}{R}, \frac{y}{R}\right)$$

satisfies (4.2) in B_R^* and $h_{1,R}(x,y) \leqslant (1 - 2\lambda)$ on $B_{R/2}^*$.

Step 2. Notations for the induction: Set, for any non-negative integer k,

$$R_k = \frac{R}{2} + \frac{R}{2^{k+1}}, \quad \text{and} \quad l_k = \mu^+ - \lambda\left(\frac{\omega}{4} + \frac{\omega}{2^{k+2}}\right).$$

We denote by $\tilde{B}_{R,\delta}^*$ the set $B_R \times (0,\delta)$ and introduce the cylinders

$$Q_k(\theta_0) = B_{\overline{R}_k} \times \left(-\theta_0^{-\alpha} R_k^{2\sigma}, 0\right) \quad \text{and} \quad \tilde{Q}_{R_k,\frac{\delta k}{4}}^*(\theta_0) = \tilde{B}_{R_k,\frac{\delta k}{4}}^* \times \left(-\theta_0^{-\alpha} R_k^{2\sigma}, 0\right).$$

We also denote

$$\begin{aligned} B(l_k, R_k) &= \left\{(x,t) \in Q_{R_k}(\theta_0);\ w(x,t) > l_k\right\}, \\ B^*(l_k, R_k) &= \left\{(x,y,t) \in Q_{R_k}^*(\theta_0);\ w^*(x,y,t) > l_k\right\} \end{aligned}$$

and

$$\tilde{B}^*\left(l_k, R_k, \frac{\delta^k}{4}\right) = \left\{(x, y, t) \in \tilde{Q}^*_{R_k, \frac{\delta^k}{4}}(\omega);\ w^*(x, y, t) > l_k\right\}.$$

From above, we define:

$$\left|B(l_k, R_k)\right| = \int_{-\theta_0^{-\alpha} R_k^{2\sigma}}^{0} \left|\left\{x \in B_{R_k};\ w(x, t) > l_k\right\}\right| dt,$$

$$\left|B^*(l_k, R_k)\right| = \int_{-\theta_0^{-\alpha} R_k^{2\sigma}}^{0} \left|\left\{(x, y) \in B^*_{R_k};\ w^*(x, y, t) > l_k\right\}\right| dt$$

and

$$\left|\tilde{B}^*\left(l_k, R_k, \frac{\delta^k}{4}\right)\right| = \int_{-\theta_0^{-\alpha} R_k^{2\sigma}}^{0} \left|\left\{(x, y) \in \tilde{B}^*_{R_k, \frac{\delta^k}{4}};\ w^*(x, y, t) > l_k\right\}\right| dt.$$

Step 3. Claim for the induction: We set

$$w_k = (w - l_k)_+, \qquad w_k^* = \left(w^* - l_k\right)_+.$$

Note that $w_k^* \neq (w_k)^*$. We consider a cut-off function $\eta_k(x, t)$ such that

$$\begin{cases} 0 < \eta_k \leqslant 1 \quad \text{in } Q_{R_k}(\theta_0), \\ \eta_k = 1 \quad \text{in } Q_{R_{k+1}}(\theta_0), \\ \eta_k = 0 \quad \text{on the parabolic boundary of } Q_{R_k}(\theta_0), \\ |\nabla \eta_k| \leqslant \dfrac{2^{k+2}}{R}, \quad (\eta_k)_t \leqslant \dfrac{2^{2\sigma(k+2)}\theta_0^\alpha}{R^{2\sigma}}, \quad \theta_0 = \dfrac{\omega}{A}. \end{cases} \tag{4.3}$$

We will use the energy inequalities of Lemma 3.4 written over the cylinders $Q^*_{R_k}(\theta_0)$, for the function $w_k^* = (w^* - l_k)_+$, where $k = 0, 1, 2, \ldots$. Let

$$Z_k = \theta_0^\alpha \left|B(l_k, R_k)\right| = \left(\frac{\omega}{A}\right)^\alpha \left|B(l_k, R_k)\right|,$$

then we are going to prove simultaneously that for every $k \geqslant 0$

$$Z_k \leqslant N^{-k} \tag{4.4}$$

for some constant $N > 1$ and

$$\eta_k w_k^* \text{ is supported in } 0 \leqslant y \leqslant \frac{\delta^k}{4}. \tag{4.5}$$

Step 4. The contraction property of the support in y direction: We first want to show that $\eta_k v_k^*$ is supported in $0 \leqslant y \leqslant \frac{\delta^k}{4}$. By a comparison principle, we have:

$$\left(w^* - \left(\mu_+ - \frac{\omega}{2}\right)\right)_+ \leqslant \left[\left(w - \left(\mu_+ - \frac{\omega}{2}\right)\right)_+ 1_{B_R}\right] * P(y) + \frac{\omega}{2} h_{1,R}(x, y)$$

in $B_R^* \times \mathbb{R}^+$, where $P(y)$ is the Poisson kernel introduced in Section 2.4 in [5]. Indeed, the right-hand side function has the trace on the boundary which is bigger than the one of the left-hand side. Moreover:

$$\begin{aligned}\left\| \left[\left(w - \left(\mu_+ - \frac{\omega}{2}\right)\right)_+ 1_{B_R}\right] * P(y)\right\|_{L^\infty(y \geqslant \frac{R}{4})} &\leqslant C \left\| P\left(\frac{R}{4}\right)\right\|_{L^2} \left(\frac{\omega}{2}\right)\left(\rho \left|Q_R(\theta_0)\right|\right)^{\frac{1}{2}} \\ &\leqslant C\sqrt{\rho}.\end{aligned}$$

Choosing ρ small enough such that this constant is smaller than $\frac{\lambda\omega}{2}$ gives:

$$\left(w^* - \left(\mu_+ - \frac{\omega}{2}\right)\right)_+ \leqslant \frac{(1-\lambda)\omega}{2},$$

in $\{y > \frac{R}{4}\} \cap Q^*_{\frac{R}{2}}(\theta_0)$. Hence

$$\left(w^* - \left(\mu_+ - \frac{\lambda\omega}{2}\right)\right)_+ = 0$$

in $\{y > \frac{R}{4}\} \cap Q^*_{\frac{R}{2}}(\theta_0)$. Since $l_0 = \mu_+ - \frac{\lambda\omega}{2}$, we obtain that $\eta_0 w_0^* = \eta_1 (v^* - l_1)_-$ is supported in $0 < y < \frac{\delta^0}{4} R = \frac{R}{4}$ for $\delta_0 = 1$.

Now we assume (4.5) is true at kth-step. We want to show that (4.5) is verified at $(k+1)$. First, we will control v_{k+1}^* in terms of $\eta_k v_k$ with some controllable error, i.e., we will show also that the following is verified at k:

$$\eta_{k+1} w_{k+1}^* \leqslant \left[(\eta_k w_k) * P(z)\right]\eta_{k+1}, \quad \text{on } \tilde{B}^*_{R_k, \frac{\delta^k}{4}} \tag{4.6}$$

where

$$\tilde{B}^*_{R_k, \frac{\delta^k}{4}} = B_{R_k} \times \left(0, \frac{\delta^k}{4}\right).$$

We consider now h_2 harmonic function defined by

$$\begin{gathered}\nabla\left(y^a \nabla h_2(z, y)\right) = 0 \quad \text{in } [0, \infty) \times [0, 1], \\ h_2(0, y) = 1, \quad 0 \leqslant y \leqslant 1, \\ h_2(z, 0) = h_2(z, 1) = 0 \quad \text{for } 0 < x < \infty.\end{gathered}$$

Then, there exists $C > 0$ such that

$$\left|h_2(z)\right| \leqslant Ce^{-\frac{z}{2}}.$$

Indeed, we can see that

$$h_2(z, y) \leqslant 2\sqrt{2}\cos\left(\frac{y}{2}\right)e^{-\frac{z}{2}} \quad (a \geqslant 0),$$
$$h_2(z, y) \leqslant 4\sqrt{2}\sin\left(\frac{y}{2}+\frac{\pi}{6}\right)e^{-\frac{z}{2}} \quad (a < 0),$$

since this function is super-harmonic and bigger than h_2 on the boundary. Consider

$$B_{\frac{R_0}{2}+\frac{R_0}{2^{k+1+\frac{1}{2}}}} \times \left[0, \frac{\delta^k}{4}\right].$$

On $y = \frac{\delta^k}{4}$ we have no contribution thanks to the induction property (4.5). The contribution of the side $y = 0$ can be controlled by $(\eta_k w_k) * P(y)$. On each of the other side, we control the contribution by the function of

$$\frac{\omega}{2}\left[h_2\left(\frac{4(x_i - \vartheta^+)}{\delta^k}, \frac{4y}{\delta^k}\right) + h_2\left(\frac{-4(x_i + \vartheta^+)}{\delta^k}, \frac{4y}{\delta^k}\right)\right]$$

where

$$\vartheta^+ = \frac{R_0}{2} + \frac{R_0}{2^{k+1+\frac{1}{2}}}.$$

Since h_2 is super-solution and on the side $x_i = \vartheta^+$ and $x_i = -\vartheta^+$ it is bigger than 1, we have, by the maximum principle:

$$w_k^* \leqslant (\eta_k w_k) * P(y) + \frac{\omega}{2}\sum_{i=1}^{n}\left[h_2\left(\frac{4(x_i - \vartheta^+)}{\delta^k}, \frac{4y}{\delta^k}\right) + h_2\left(\frac{-4(x_i + \vartheta^+)}{\delta^k}, \frac{4y}{\delta^k}\right)\right].$$

For $x \in B_{R_{k+1}}$,

$$\frac{\omega}{2}\sum_{i=1}^{n}\left[h_2\left(\frac{4(x_i - \vartheta^+)}{\delta^k}, \frac{4y}{\delta^k}\right) + h_2\left(\frac{-4(x_i + \vartheta^+)}{\delta^k}, \frac{4y}{\delta^k}\right)\right] \leqslant 2nC\omega e^{-\frac{\sqrt{2}-1}{2^{k+3}\delta^k}} \leqslant \lambda 2^{-k-4},$$

for some small constant δ and λ. This gives (4.6) since

$$w_{k+1}^* \leqslant \left(w_k^* - \lambda 2^{-k-3}\right)_+.$$

More precisely, this gives

$$w_{k+1}^* \leqslant \left((\eta_k w_k) * P(y) - \lambda 2^{-k-4}\right)_+ \quad \text{and} \quad \eta_{k+1} w_{k+1}^* \leqslant \left((\eta_k w_k) * P(y) - \lambda 2^{-k-4}\right)_+.$$

Then, we find for $\frac{\delta^{k+1}}{4} \leqslant y \leqslant \frac{\delta^k}{4}$,

$$\left|(\eta_k w_k) * P(y)\right| \leqslant \sqrt{Z_k}\left\|P(y)\right\|_{L^2} \leqslant \frac{\sqrt{\frac{\omega}{2}}N^{-\frac{k}{2}}}{(\frac{\delta}{4})^{\frac{(k+1)n}{2}}}\left\|P(1)\right\|_{L^2} \leqslant \lambda 2^{-k-4}$$

for large $N > \sup(\frac{4^{n+1}}{\delta^2}, \frac{2^{4n+9}\omega\|P(1)\|_{L^2}^2}{\lambda^2\delta^{2n}})$. Therefore,

$$\eta_{k+1} w_{k+1}^* \leqslant 0 \quad \text{for } \frac{\delta^{k+1}}{4} \leqslant y \leqslant \frac{\delta^k}{4}.$$

Step 5. Local Energy Estimate: First, we find the lower bound of the following quantity

$$D = \int_{B_r \times \{t_2\}} \eta^2 \left[\int_0^{(w-k)_+} (M-k-\xi)^{\frac{1}{m}-1}\xi\, d\xi\right] dx.$$

Let's define the function $F(\xi)$ by

$$F(\xi) = (M-k-\xi)^{\frac{1}{m}-1}\xi = (M-k-\xi)^{-\alpha}\xi \quad (0 \leqslant \xi \leqslant M-k).$$

Then, we have

$$F'(\xi) = \frac{1}{m}(M-k-\xi)^{-\alpha-1}\big(m(M-k)-\xi\big),$$
$$F''(\xi) = -\frac{1}{m}\left(\frac{1}{m}-1\right)(M-k-\xi)^{-\alpha-2}\big(2m(M-k)-\xi\big).$$

Note that the sign change of second derivatives of $F(\xi)$ takes place at $2m(M-k)$. If $(w-k)_+$ is smaller than $2m(M-k)$, we get

$$\int_0^{(w-k)_+} F(\xi)\, d\xi \geqslant \frac{1}{2}(w-k)_+ F\left(\frac{(w-k)_+}{2}\right)$$

and

$$D \geqslant \frac{1}{4}\left(\frac{M-k}{2}\right)^{-\alpha} \int_{B_r \times \{t_2\}} \left[\eta(w-k)_+\right]^2 dx,$$

since $F''(\xi) \leqslant 0$ $(0 \leqslant \xi \leqslant (w-k)_+)$. On the other hand,

$$\int_0^{(w-k)_+} F(\xi)\, d\xi \geqslant \frac{1}{2} \cdot 2m(M-k)F\big(2m(M-k)\big) = 2m^2(M-k)^2\big[(1-2m)(M-k)\big]^{-\alpha}$$
$$\geqslant 2m^2(w-k)_+^2\big[(1-2m)(M-k)\big]^{-\alpha}$$

when $(w-k)_+ \geqslant 2m(M-k)$. Thus we obtain

$$D \geqslant 2\big[(1-2m)(M-k)\big]^{-\alpha} \int_{B_r \times \{t_2\}} \big[\eta(w-k)_+\big]^2\,dx.$$

Therefore, there is a small constant $c>0$ such that

$$c(M-k)^{-\alpha} \int_{B_r \times \{t_2\}} \big[\eta(w-k)_+\big]^2\,dx \leqslant \int_{B_r \times \{t_2\}} \eta^2 \left[\int_0^{(w-k)_+} (M-k-\xi)^{\frac{1}{m}-1} \xi\,d\xi \right] dx.$$

Next, notice that from Step 4, then we have

$$\begin{aligned}
\big\|\eta_k w_k^*\big\|^2_{L^2(B^*_{R_k}, |y|^a)} &\leqslant \big\|(\eta_{k-1} w_{k-1}) * P(y)\big\|^2_{L^2(B^*_{R_{k-1}}, |y|^a)} \\
&\leqslant \int_0^{\delta^{k-1}/4} \big\|(\eta_{k-1} w_{k-1}) * \big(P(y)\big)\big\|^2_{L^2(B_{R_{k-1}})} y^a\,dy \\
&\leqslant \big\|P(1)\big\|^2_{L^1(\mathbb{R}^n)} \big\|(\eta_{k-1} w_{k-1})\big\|^2_{L^2(B_{R_{k-1}})} \int_0^{\delta^{k-1}/4} y^a\,dy \\
&\leqslant \frac{1}{1+a}\left(\frac{\delta^{k-1}}{4}\right)^{a+1} \big\|P(1)\big\|^2_{L^1(B_{R_{k-1}})} \|\eta_{k-1} w_{k-1}\|^2_{L^2(B_{R_{k-1}})} \\
&\leqslant \frac{1}{1+a}\left(\frac{\delta^{k-1}}{4}\right)^{a+1} \frac{(\lambda\omega)^2}{4} \big|A_{l_{k-1}, R_{k-1}}(t)\big|, \qquad (4.7)
\end{aligned}$$

where $A_{l,R}(t) = \{x \in B_R;\ w(x,t) > l\}$. We can apply Lemma 3.4 (Local Energy Estimate) on $\eta_k w_k^* 1_{\{0<y<\frac{\delta^k}{4}\}}$

$$\begin{aligned}
&c\left(\frac{\lambda\omega}{4}\right)^{-\alpha} \sup_{-\theta_0^{-\alpha} R_k^{2\sigma} < t < 0} \|\eta_k w_k\|^2_{L^2(B_{R_k})} + \big\|\nabla\big(\eta_k w_k^*\big)\big\|^2_{L^2(Q^*_{R_k}(\theta_0), |y|^a)} \\
&\quad \leqslant \frac{(\delta^{k-1})^{a+1} 4^{k-a} (\lambda\omega)^2}{(1+a)R^2} \int_{-\theta_0^{-\alpha} R_k^{2\sigma}}^0 \big|A_{l_{k-1}, R_{k-1}}(t)\big|\,dt \\
&\qquad + \frac{4^{\sigma(k+2)} \theta_0^\alpha (\lambda\omega)^2}{2m M^\alpha R^{2\sigma}} \int_{-\theta_0^{-\alpha} R_k^{2\sigma}}^0 \big|A_{l_k, R_k}(t)\big|\,dt. \qquad (4.8)
\end{aligned}$$

From this, it follows that

$$c\left(\frac{4}{A\lambda}\right)^{\alpha} \sup_{-\theta_0^{-\alpha} R_k^{2\sigma} < t < 0} \|\eta_k w_k\|^2_{L^2(B_{R_k})} + \theta_0^{\alpha} \left\|\nabla\left(\eta_k w_k^*\right)\right\|^2_{L^2(Q^*_{R_k}(\theta_0), |y|^a)}$$
$$\leqslant (\lambda\omega)^2 \left(\frac{(\delta^{k-1})^{a+1} 4^{k-a}}{(1+a)R^2} + \frac{4^{\sigma(k+2)}\theta_0^{\alpha}}{2mM^{\alpha}R^{2\sigma}}\right) Z_{k-1}. \tag{4.9}$$

Now let us make a change of variable

$$\tau = \left(\frac{\omega}{A}\right)^{\alpha} t = \theta_0^{\alpha} t.$$

Then, $Q_R(\theta_0)$, $Q_R^*(\theta_0)$ and $\tilde{Q}_R^*(\theta_0)$ will be transformed to $Q_R(1)$, $Q_R^*(1)$ and $\tilde{Q}_R^*(1)$ respectively. Let $\overline{w}(x,\tau) = w(x, \theta_0^{-\alpha}\tau)$ and $\overline{w}^*(x,y,\tau) = w^*(x,y,\theta_0^{-\alpha}\tau)$. We also define the quantity $\overline{Z}_k$ to be

$$\overline{Z}_k = \left|\left\{(x,t) \in Q_{R_k}(1):\ \overline{w} > l_k\right\}\right|.$$

Then, $\overline{Z}_k$ will be equal to Z_k. After the change of variable, we have

$$c\left(\frac{4}{A\lambda}\right)^{\alpha} \sup_{-R_k^{2\sigma} < \tau < 0} \|\eta_k \overline{w}_k\|^2_{L^2(B_{R_k})} + \left\|\nabla\left(\eta_k \overline{w}_k^*\right)\right\|^2_{L^2(Q^*_{R_k}(1), |y|^a)}$$
$$\leqslant (\lambda\omega)^2 \left(\frac{(\delta^{k-1})^{a+1} 4^{k-a}}{(1+a)R^2} + \frac{4^{\sigma(k+2)}\theta_0^{\alpha}}{2mM^{\alpha}R^{2\sigma}}\right) \overline{Z}_{k-1}.$$

Let's choose a small constant A such that $c(\frac{4}{A\lambda})^{\alpha} > 1$. Then

$$\sup_{-R_k^{2\sigma} < \tau < 0} \|\eta_k \overline{w}_k\|^2_{L^2(B_{R_k})} + \left\|\nabla\left(\eta_k \overline{w}_k^*\right)\right\|^2_{L^2(Q^*_{R_k}(1), |y|^a)}$$
$$\leqslant (\lambda\omega)^2 \left(\frac{(\delta^{k-1})^{a+1} 4^{k-a}}{(1+a)R^2} + \frac{4^{\sigma(k+2)}\theta_0^{\alpha}}{2mM^{\alpha}R^{2\sigma}}\right) \overline{Z}_{k-1}. \tag{4.10}$$

Since $\eta_k \overline{w}_k^* 1_{\{0<y<\frac{\delta^{k-1}}{4}\}}$ has the same trace at $y=0$ as $(\eta_k \overline{w}_k)^*$, we have

$$C \int_0^{\frac{\delta^{k-1}}{4}} \int_{\mathbb{R}^n} \left|\nabla\left(\eta_k \overline{w}_k^*\right)\right|^2 y^a\,dx\,dy = \int_0^{\infty} \int_{\mathbb{R}^n} \left|\nabla\left(\eta_k \overline{w}_k^* 1_{\{0<y<\frac{\delta^{k-1}}{4}\}}\right)\right|^2 y^a\,dx\,dy$$
$$\geqslant \int_0^{\infty} \int_{\mathbb{R}^n} \left|\nabla(\eta_k \overline{w}_k)^*\right|^2 y^a\,dx\,dy.$$

We also have

$$\int_{Q_{R_k}(1)} |\eta_k \overline{w}_k|^2 \,dx\,d\tau \geqslant (l_{k+1}-l_k)^2 \int_{-R_k^{2\sigma}}^{0} \big|\big\{(x,t)\in Q_{R_{k+1}}(1)\colon\ \overline{w} > l_{k+1}\big\}\big|\,d\tau$$
$$= \left(\frac{\lambda\omega}{2^{k+3}}\right)^2 \overline{Z}_{k+1}.$$

Hence, combining above estimates with Sobolev inequalities (Lemma 3.3), the inequality (4.10) changes into

$$\begin{aligned}\left(\frac{\lambda\,\omega}{2^{k+3}}\right)^2 \overline{Z}_{k+1} &\leqslant \|\eta_k \overline{w}_k\|^2_{L^2(Q_{R_k}(1))}\\ &\leqslant C\Big[\sup_{-R_k^{2\sigma}\leqslant t\leqslant 0} \|\eta_k\, w_k\|^2_{L^2(\mathbb{R}^n)} + \big\|\nabla\big(\eta_k \overline{w}_k^*\big)\big\|^2_{L^2(Q^*_{R_k}(1),|y|^a)}\Big]\,\overline{Z}_{k-1}^{\frac{2\sigma}{n+2\sigma}}\\ &\leqslant C(\lambda\omega)^2\left(\frac{(\delta^{k-1})^{a+1}4^{k-a}}{(1+a)R^2} + \frac{4^{\sigma(k+2)}\theta_0^\alpha}{2mM^\alpha R^{2\sigma}}\right)\overline{Z}_{k-1}^{1+\frac{2\sigma}{n+2\sigma}}\end{aligned}$$

for some constant $C>0$. Since $0<\sigma,\delta<1$, we have

$$\overline{Z}_{k+1} \leqslant C4^{2k}\left(\frac{4^{3-a}}{(1+a)R^2} + \frac{4^5\theta_0^\alpha}{2mM^\alpha R^{2\sigma}}\right)\overline{Z}_{k-1}^{1+\frac{2\sigma}{n+2\sigma}} = C'4^{2k}\overline{Z}_{k-1}^{1+\frac{2\sigma}{n+2\sigma}}.$$

Let's choose the constant N to satisfy

$$N > \sup\left(1, C', 16^{\frac{n+2\sigma}{\sigma}}, \frac{4^{n+1}}{\delta^2}, \frac{2^{4n+9}\omega\|P(1)\|^2_{L^2}}{\lambda^2\delta^{2n}}\right).$$

Then

$$\left(\frac{N}{16^{\frac{n+2\sigma}{2\sigma}}}\right)^{\frac{2\sigma k}{n+2\sigma}} \geqslant N^{\frac{\sigma k}{n+2\sigma}} \geqslant N^4 \geqslant C'N^{2(1+\frac{2\sigma}{n+2\sigma})}$$

for $k \geqslant \frac{4(n+2\sigma)}{\sigma}$ and this is equivalent to $N^{-k} \geqslant C'4^{2k}N^{-(1+\frac{2\sigma}{n+2\sigma})(k-2)}$. If we take the constant ρ so sufficiently small that

$$\overline{Z}_{\bar{k}} \leqslant N^{-\bar{k}} \quad \left(1 \leqslant \bar{k} < \frac{4(n+2\sigma)}{\sigma} + 1\right),$$

then (4.4) is true for all $k \geqslant 0$. □

To deal with alternative case, we follow the details in [9]. We assume that the assumption of Lemma 4.1 is violated, i.e., for every sub-cylinder $Q_R(\theta_0)$

$$\left|\left\{(x,t)\in Q_R(\theta_0)\colon\ w(x,t)>\mu^+-\frac{\omega}{2}\right\}\right|>\rho\left|Q_R(\theta_0)\right|.$$

Since

$$\mu^+-\frac{\omega}{2}\geqslant\mu^-+\frac{\omega}{2^{s_0}},\qquad\forall s_0\geqslant 2,$$

we rewrite this as

$$\left|\left\{(x,t)\in Q_R(\theta_0)\colon\ w(x,t)\leqslant\mu^-+\frac{\omega}{2^{s_0}}\right\}\right|\leqslant(1-\rho)\left|Q_R(\theta_0)\right| \tag{4.11}$$

valid for all cylinders $Q_R(\theta_0)$.

Lemma 4.2. *If* (4.1) *is violated, then there exists a time level*

$$t^*\in\left[-\theta_0^{-\alpha}R^{2\sigma},-\frac{\rho}{2}\theta_0^{-\alpha}R^{2\sigma}\right]$$

such that

$$\left|\left\{x\in B_R;\ w(x,t)\leqslant\mu^-+\frac{\omega}{2^{s_0}}\right\}\right|\leqslant\frac{1-\rho}{1-\frac{\rho}{2}}|B_R|.$$

Proof. If not, for all $t\in[-\theta_0^{-\alpha}R^{2\sigma},-\frac{\rho}{2}\theta_0^{-\alpha}R^{2\sigma}]$,

$$\left|\left\{x\in B_R;\ w(x,t)\leqslant\mu^-+\frac{\omega}{2^{s_0}}\right\}\right|>\frac{1-\rho}{1-\frac{\rho}{2}}|B_R|$$

and

$$\begin{aligned}\left|\left\{(x,t)\in Q_R(\theta_0)\colon\ w(x,t)\leqslant\mu^-+\frac{\omega}{2^{s_0}}\right\}\right|&\geqslant\int_{-\theta_0^{\alpha}R^{2\sigma}}^{-\frac{\rho}{2}\theta_0^{\alpha}R^{2\sigma}}\left|\left\{x\in B_R;\ w(x,\tau)\leqslant\mu^-+\frac{\omega}{2^{s_0}}\right\}\right|d\tau\\&>(1-\rho)\left|Q_R^*(\theta_0)\right|,\end{aligned}$$

contradicting (4.11). □

The lemma asserts that at some time level t^* the set where w is close to its supremum occupies only a portion of the B_R. The next lemma claims that this indeed occurs for all time levels near the $Q_R(\theta_0)$. Set

$$H=\sup_{B_R^*\times[t^*,0]}\left|\left(w^*-\left(\mu^-+\frac{\omega}{2^{s_0}}\right)\right)_-\right|.$$

Lemma 4.3. *There exists a positive integer $s_1 > s_0$ such that if*

$$H > \frac{\omega}{2^{s_1}}, \tag{4.12}$$

then

$$\left|\left\{x \in B_R;\ w(x,t) \leqslant \mu^- + \frac{\omega}{2^{s_1}}\right\}\right| \leqslant \left(1 - \left(\frac{\rho}{2}\right)^2\right)|B_R|, \quad \forall t \in \left[t^*, 0\right].$$

Proof. We introduce the logarithmic function which appears in Section 2 in [7]

$$\Psi\left(H, \left(w^* - k\right)_-, c\right) \equiv \max\left\{\ln\left(\frac{H}{H - (w^* - k)_- + c}\right); 0\right\}$$

for $k = \mu^- + \frac{\omega}{2^{s_0}}$, $c = \frac{\omega}{2^{s_1}}$. From the definition and the indicated choices we have

$$\Psi\left(H, \left(w^* - k\right)_-, c\right) \leqslant (s_1 - s_0)\ln 2 \quad \text{and} \quad \left|\Psi_{w^*}\left(H, \left(w^* - k\right)_-, c\right)\right|^2 \leqslant \left(\frac{2^{s_1}}{\omega}\right)^2.$$

To simplify the symbolism, let us set $\Psi(H, (w^* - k)_-, c) = \varphi(w^*)$. We apply to (1.4) the testing function

$$\left(M - w^*\right)^\alpha \frac{\partial}{\partial w^*}\left[\varphi^2\left(w^*\right)\right]\zeta^2 = \left(M - w^*\right)^\alpha \left[\varphi^2\left(w^*\right)\right]'\zeta^2$$

where $\zeta(x, z)$ is the smooth cut-off function such that

$$\zeta = 1 \quad \text{in } B_{(1-\nu)R} \times \left(-(1-\nu)R, (1-\nu)R\right), \qquad \zeta = 0 \quad \text{on } \partial\left\{B_R \times (-R, R)\right\}$$

and

$$|D\zeta| \leqslant \frac{2}{\nu R}.$$

Then, we have for every $t^* < t < t_0$

$$\begin{aligned}
0 &= -\int_{B_R^*} y^a \nabla\left[\left(M - w^*\right)^\alpha \left(\varphi^2\right)'\zeta^2\right] \cdot \nabla w^* \, dx\, dy \\
&\leqslant 2\alpha \int_{B_R^*} y^a \varphi\varphi'\zeta^2\left(M - w^*\right)^{\alpha-1}\left|\nabla w^*\right|^2 dx\, dy + 2\int_{B_R^*} y^a\left(M - w^*\right)^\alpha \varphi|\nabla\zeta|^2\, dx\, dy \\
&\quad - 2\int_{B_R^*} y^a\left(M - w^*\right)^\alpha \zeta^2\left(\varphi'\right)^2\left|\nabla w^*\right|^2 dx\, dy - \frac{1}{m}\int_{B_R} \zeta^2\left(\varphi(w)^2\right)' w_t\, dx.
\end{aligned}$$

Since φ vanishes on the set where $(w^* - k)_- = 0$, we have

$$\alpha \int_{B_R^*} y^a \varphi\varphi' \zeta^2 \big(M - w^*\big)^{\alpha-1} \big|\nabla w^*\big|^2 \, dx\, dy \leqslant \int_{B_R^*} y^a \big(M - w^*\big)^{\alpha} \zeta^2 \big(\varphi'\big)^2 \big|\nabla w^*\big|^2 \, dx\, dy$$

when $-\frac{\alpha}{e} + 1 \leqslant 2^{s_0}$. Hence

$$\begin{aligned}
&\sup_{t^*<t<0} \int_{B_R} \Psi^2\big(H, (w-k)_-, c\big)(x,0,t)\zeta^2(x,0)\, dx \\
&\quad \leqslant \int_{B_R} \Psi^2\big(H, (w-k)_-, c\big)\big(x,0,t^*\big)\zeta^2(x,0)\, dx \\
&\qquad + 2m \int_{B_R^* \times [t^*,0]} y^a \big(M - w^*\big)^{\alpha} \Psi\big(H, \big(w^* - k\big)_-, c\big) |\nabla \zeta|^2 \, dx\, dy\, d\tau.
\end{aligned}$$

Combining this with the previous Lemma 4.2 gives

$$\begin{aligned}
&\int_{B_R} \Psi^2\big(H, (w-k)_-, c\big)(x,0,t)\zeta^2(x,0)\, dx \\
&\quad \leqslant \big[(s_1 - s_0)\ln 2\big]^2 \left(\frac{1-\rho}{1-\frac{\rho}{2}}\right) |B_R| + \frac{R^{2\sigma+a-1}\omega^{\alpha}(1 - \frac{1}{2^{s_0}})^{\alpha}}{(1+a)\nu^2} (s_1 - s_0)\ln 2 |B_R|. \qquad (4.13)
\end{aligned}$$

The integral on the left-hand side of (4.13) is estimated below by integrating over the smaller set

$$\left\{ x \in B_{(1-\nu)R}\colon\ w(x,t) < \mu^- + \frac{\omega}{2^{s_1}} \right\}.$$

On such a set, since the function Ψ is a decreasing function of H, we find

$$\Psi^2\left(H, \left(w - \left(\mu^- + \frac{\omega}{2^{s_0}}\right)\right)_-, \frac{\omega}{2^{s_1}}\right) \geqslant (s_1 - s_0 - 1)^2 \ln^2 2,$$

and therefore (4.13) gives

$$\begin{aligned}
&\left|\left\{ x \in B_{(1-\nu)R}\colon\ w(x,t) < \mu^- + \frac{\omega}{2^{s_1}} \right\}\right| \\
&\quad \leqslant \left(\frac{1-\rho}{1-\frac{\rho}{2}}\right) \left(\frac{s_1 - s_0}{s_1 - s_0 - 1}\right)^2 |B_R| + \frac{\gamma}{\nu^2(s_1 - s_0 - 2)} |B_R|
\end{aligned}$$

for constant $\gamma = \frac{R^{2\sigma+a-1}\omega^{\alpha}(1-\frac{1}{2^{s_0}})^{\alpha}}{(1+a)\ln 2}$. On the other hand

$$\begin{aligned}
&\left|\left\{x \in B_R\colon\ w(x,t) < \mu^- + \frac{\omega}{2^{s_1}}\right\}\right| \\
&\quad \leqslant \left|\left\{x \in B_{(1-\nu)R}\colon\ w(x,t) < \mu^- + \frac{\omega}{2^{s_1}}\right\}\right| + |B_R \setminus B_{(1-\nu)R}| \\
&\quad \leqslant \left|\left\{x \in B_{(1-\nu)R}\colon\ w(x,t) < \mu^- + \frac{\omega}{2^{s_1}}\right\}\right| + n\nu|B_R|.
\end{aligned}$$

Therefore

$$\left|\left\{x \in B_R\colon\ w(x,t) < \mu^- + \frac{\omega}{2^{s_1}}\right\}\right| \leqslant \left[\left(\frac{1-\rho}{1-\frac{\rho}{2}}\right)\left(\frac{s_1-s_0}{s_1-s_0-1}\right)^2 + \frac{\gamma}{\nu^2(s_1-s_0-2)} + n\nu\right]|B_R|$$

for all $t \in (t^*, 0)$.

To prove the lemma, we first choose ν so small that $n\nu \leqslant \frac{3}{8}\rho^2$ and then s_1 so large that

$$\frac{\gamma}{\nu^2(s_1-s_0-2)} \leqslant \frac{3}{8}\rho^2, \qquad \left(\frac{s_1-s_0}{s_1-s_0-1}\right)^2 \leqslant \left(1-\frac{1}{2}\rho\right)(1+\rho). \qquad \square$$

Well-definedness of $\lim_{y\to 0} y^a w_y^*$ gives us that there is no jumping near the hyper-plane $y=0$. More precisely,

$$w^*(x,y,t) - w(x,t) = O\big(y^{1-a}\big) \quad \text{a.e. } B_R$$

as y goes to zero since the fact that $y^a w_y^*$ has a limit as $y \to 0$, immediately implies that $\lim_{y\to 0} \frac{w^*(x,y,t)-w(x,t)}{y^{1-a}}$ has the same limit. Hence, we can also make similar estimates of w^* in the cube B_R^* using the measure condition (Lemma 4.3):

$$\left|\left\{x \in B_R;\ w(x,t) > \mu^- + \frac{\omega}{2^{s_1}}\right\}\right| > \left(\frac{\rho}{2}\right)^2 |B_R| > 0$$

for all $t \in [t^*, 0]$. The conclusion follows.

Lemma 4.4. *There exists a constant $\overline{\rho} > 0$ such that*

$$\left|\left\{(x,y) \in B_R^*\colon\ w^*(x,y,t) > \mu^- + \frac{\omega}{2^{s_1+1}}\right\}\right| > \overline{\rho}\big|B_R^*\big|$$

for all $t \in [t^*, t_0]$.

We list lemma from the literature and adapted here to our situation.

Lemma 4.5. *(See De Giorgi [8].) If $f \in W^{1,1}(B_r)$ $(B_r \subset \mathbb{R}^n)$ and $l, k \in \mathbb{R}$, $k < l$, then*

$$(l-k)\left|\left\{x \in B_r\colon\ f(x) > l\right\}\right| \leqslant \frac{C r^{n+1}}{|\{x \in B_r\colon\ f(x) < k\}|} \int\limits_{k<f<l} |\nabla f|\, dx,$$

where C depends only on n.

For the remainder of this section we assume that (4.12) holds.

Lemma 4.6. *If* (4.1) *is violated, for every $\nu_* \in (0,1)$, there exists a number $s^* > s_1 + 1 > s_0$ independent of ω and R such that*

$$\left|\left\{(x,y,t) \in Q_R^*\left(\theta_0, 0, 1-\frac{\rho}{2}\right)\colon\ w^*(x,y,t) \leqslant \mu^- + \frac{\omega}{2^{s^*}}\right\}\right| \leqslant \nu_* \left|Q_R^*\left(\theta_0, 0, 1-\frac{\rho}{2}\right)\right|.$$

Proof. Apply Lemma 4.5 over the cube B_R^* for $f(x,y) = -w^*(x,y,t)$, $t \in (-\frac{\rho\theta_0^{-\alpha}R^{2\sigma}}{2}, 0)$ and the levels

$$l = -\mu^- - \frac{\omega}{2^{s+1}}, \qquad k = -\mu^- - \frac{\omega}{2^s}, \quad s = s_1+1, s_1+2, \ldots, s^*.$$

Then, from Lemma 4.4, we have

$$\left(\frac{\omega}{2^{s+1}}\right)\left|A_{s+1}(t)\right| \leqslant \frac{4CR^{n+2}}{\bar{\rho}|B_R^*|} \int\limits_{A_s(t)\setminus A_{s+1}(t)} \left|\nabla w^*\right| dx\, dy \tag{4.14}$$

where

$$A_s(t) = \left\{(x,y) \in B_R^*\colon\ w^*(x,y,t) < \mu^- + \frac{\omega}{2^s}\right\}.$$

Set

$$A_s = \left\{(x,y,t) \in B_R^* \times \left[-\frac{\rho\theta_0^{-\alpha}R^{2\sigma}}{2}, 0\right]\colon\ w^*(x,y,t) < \mu^- + \frac{\omega}{2^s}\right\}.$$

From this, integrating (4.14) over $(-\frac{\rho\theta_0^{-\alpha}R^{2\sigma}}{2}, 0)$ we get

$$\begin{aligned}\left(\frac{\omega}{2^{s+1}}\right)|A_{s+1}| &\leqslant C'R \iint\limits_{A_s\setminus A_{s+1}} \left|\nabla w^*\right| y^{\frac{a}{2}} y^{-\frac{a}{2}}\, dx\, dy\, d\tau \\ &\leqslant C'R\left(\iint\limits_{Q_R^*(\theta_0,0,1-\frac{\rho}{2})} \left|\nabla w^*\right|^2 y^a\, dx\, dy\, d\tau\right)^{\frac{1}{2}}\end{aligned}$$

$$\times\left(\iint\limits_{Q_R^*(\theta_0,0,1-\frac{\rho}{2})}\left(y^{-\frac{a}{2}}\right)^{\frac{4}{1+a}}\,dx\,dy\,d\tau\right)^{\frac{1+a}{4}}$$
$$\times\left(\iint\limits_{A_s\setminus A_{s+1}}dx\,dy\,d\tau\right)^{\frac{1-a}{4}}. \tag{4.15}$$

Take the $\frac{4}{1-a}$th power to obtain

$$\left(\frac{\omega}{2^{s+1}}\right)^{\frac{4}{1-a}}|A_{s+1}|^{\frac{4}{1-a}}\leqslant C''\big(|A_s|-|A_{s+1}|\big)\left(\iint\limits_{Q_R^*(\theta_0,0,1-\frac{\rho}{2})}\left|\nabla w^*\right|^2y^a\,dx\,dy\,d\tau\right)^{\frac{2}{1-a}} \tag{4.16}$$

where $C''=(C'R)^{\frac{4}{1-a}}(\frac{1+a}{1-a}R^{n+\frac{1-a}{1+a}}(t_0-t^*))^{\frac{1+a}{1-a}}$.

We estimate the integral on the right-hand side by making use of (3.5) written over again as $Q_{2R}^*(\theta_0)$, $k=\mu^-+\frac{\omega}{2^s}$ and as $\eta(x,z,t)$ a smooth cut-off function in $Q_{2R}^*(\theta_0)$ which equals one on $Q_R^*(\theta_0)$, vanishes on the parabolic boundary of $Q_{2R}^*(\theta_0)$ and is such that

$$|\nabla\eta|\leqslant\frac{1}{R},\qquad|\eta_t|\leqslant\frac{\omega^\alpha}{A^\alpha R^{2\sigma}}.$$

Then, since $|\nabla w^*|=|\nabla(w^*-k)_-|$, we deduce

$$\left[\iint\limits_{Q_R^*(\theta_0,0,1-\frac{\rho}{2})}\left|\nabla\big(w^*-k\big)_-\right|^2y^a\,dy\,dx\,d\tau\right]^{\frac{2}{1-a}}$$
$$\leqslant C(A,a,M,n,R,\sigma,\alpha)\left(\frac{\omega}{2^s}\right)^{\frac{4}{1-a}}\left|Q_R\left(\theta_0,0,1-\frac{\rho}{2}\right)\right|^{\frac{2}{1-a}}.$$

Substitute this estimate into (4.16) and divide through by $(\frac{\omega}{2^{s+1}})^{\frac{4}{1-a}}$.

$$|A_{s+1}|^{\frac{4}{1-a}}\leqslant C'''\left|Q_R^*\left(\theta_0,0,1-\frac{\rho}{2}\right)\right|^{\frac{2}{1-a}}\big(|A_s|-|A_{s+1}|\big). \tag{4.17}$$

These inequalities are valid for all $s_1+1\leqslant s\leqslant s^*-1$. Adding (4.17) for $s=s_1+1,s_1+2,\dots,s^*-1$, we have

$$\big(s^*-s_1-2\big)|A_{s^*}|^{\frac{4}{1-a}}\leqslant C'''\left|Q_R^*\left(\theta_0,0,1-\frac{\rho}{2}\right)\right|^{\frac{3-a}{1-a}}.$$

To prove the lemma, we divide (s^*-s_1-2) and take s^* so large that

$$\left(\frac{C'''}{s^*-s_1-2}\right)^{\frac{1-a}{4}}\frac{1}{|Q_R^*(\theta_0,0,1-\frac{\rho}{2})|^{\frac{1+a}{4}}}\leqslant\nu^*.\qquad\square$$

Using the relation between w and w^*, we show next that we can replace the $Q_R^*(\theta_0, 0, 1-\frac{\rho}{2})$ and w^* by $Q_R(\theta_0, 0, 1-\frac{\rho}{2})$ and w respectively.

Lemma 4.7. *In addition, we have*

$$\left|\left\{(x,t) \in Q_R\left(\theta_0, 0, 1-\frac{\rho}{2}\right):\ w(x,t) \leqslant \mu^- + \frac{\omega}{2^{s^*+1}}\right\}\right| \leqslant \nu^\star \left|Q_R\left(\theta_0, 0, 1-\frac{\rho}{2}\right)\right|$$

for every $\nu_\star \in (0,1)$.

Proof. For every t, x fixed and for $k = \mu^- + \frac{\omega}{2^{s^*}}$,

$$\begin{aligned}(w-k)_- &= \left(w^* - k\right)_-(s) - \int_0^s \partial_y\left[\left(w^*-k\right)_-\right] dy \\ &\leqslant \left(w^*-k\right)_-(s) + \frac{s^{1-a}}{1-a}\left(\int_0^s \left|\nabla\left(w^*-k\right)_-\right|^2 y^a\, dy\right)^{\frac{1}{2}}.\end{aligned}$$

So, integrating in x and t over $Q_R(\theta_0, 0, 1-\frac{\rho}{2})$, we have

$$\begin{aligned}\int\limits_{Q_R(\theta_0,0,1-\frac{\rho}{2})} \left[(w-k)_-\right]^2 dx\,dt \leqslant 2\Bigg[&\int\limits_{Q_R(\theta_0,0,1-\frac{\rho}{2})} \left[\left(w^*-k\right)_-(s)\right]^2 dx\,dt \\ &+ \frac{s^{2-2a}}{(1-a)^2} \int\limits_{Q_R^*(\theta_0,0,1-\frac{\rho}{2})} \left|\nabla\left(w^*-k\right)_-\right|^2 y^a\, dx\,dy\,dt\Bigg]\end{aligned}$$

for any $y = s \leqslant R$. Hence, integrating in s over $[0, \epsilon_1]$,

$$\begin{aligned}\epsilon_1 \int\limits_{A(t)} \left[(w-k)_-\right]^2 dx\,dt \leqslant 2\Bigg[&\int\limits_{Q_R^*(\theta_0,0,1-\frac{\rho}{2})} \left[\left(w^*-k\right)_-(s)\right]^2 dx\,dy\,dt \\ &+ \frac{{\epsilon_1}^{3-2a}}{(3-2a)(1-a)^2} \int\limits_{Q_R^*(\theta_0,0,1-\frac{\rho}{2})} \left|\nabla\left(w^*-k\right)_-\right|^2 y^a\, dx\,dy\,dt\Bigg].\end{aligned}$$

By making use of (3.5) written over as $Q_{2R}^*(\theta_0)$, $k = \mu^- + \frac{\omega}{2^s}$ and as $\eta(x,z,t)$ which is given in the proof of Lemma 4.6. Then, we obtain

$$\iint\limits_{Q_R^*(\theta_0,0,1-\frac{\rho}{2})} \left|\nabla\left(w^*-k\right)_-\right|^2 y^a\, dy\,dx\,d\tau \leqslant C\left(\frac{\omega}{2^{s^*}}\right)^2 \left|Q_R\left(\theta_0, 0, 1-\frac{\rho}{2}\right)\right|$$

for some constant $C = C(A, a, M, n, R, \sigma, \alpha)$. Thus, we have

$$\begin{aligned}
&\epsilon_1\left(\frac{\omega}{2^{s^*+1}}\right)^2\left|\left\{(x,t)\in Q_R\left(\theta_0,0,1-\frac{\rho}{2}\right):\ w\leqslant \mu^-+\frac{\omega}{2^{s^*+1}}\right\}\right| \\
&\quad\leqslant \epsilon_1\int\limits_{A(t)}\big[(w-k)_-\big]^2\,dx\,dt \\
&\quad\leqslant C\left(\frac{\omega}{2^{s^*}}\right)^2\left(\nu^*+\frac{{\epsilon_1}^{3-2a}}{(3-2a)(1-a)^2}\right)\left|Q_R\left(\theta_0,0,1-\frac{\rho}{2}\right)\right|.
\end{aligned}$$

To prove the lemma, we take ν^* and ϵ_1 so small that

$$4C\left(\frac{\nu^*}{\epsilon_1}+\frac{{\epsilon_1}^{2-2a}}{(3-2a)(1-a)^2}\right)\leqslant \nu^\star$$

for $\nu^\star$ in $(0,1)$. □

Now we show that given $\nu^\star$, it determines a level $\mu^- + \frac{\omega}{2^{s^*+1}}$ and a cylinder so that the measure of the set where w is below such a level can be made smaller than $\nu^\star$, on that particular cylinder. Hence, for sufficiently small number $\nu^\star$, we have a new powerful assumption like that of Lemma 4.1. Therefore, using the same arguments as in Lemma 4.1 with $(w-(\mu^-+\frac{\omega}{2^{s^*+1}}))_-$ we can obtain the following result.

Lemma 4.8. *The number $\nu^\star$ (and hence s^*) can be chosen so that*

$$w(x,t)\geqslant \mu^-+\frac{\lambda\omega}{2^{s^*+2}}\quad a.e.\ Q_{\frac{R}{2}}\left(\theta_0,0,\frac{\rho}{2}\right)$$

for some λ in $(0,\frac{1}{2})$.

Lemma 4.9 *(Oscillation Lemma). There exist constants $\lambda^* > 0$ and $\kappa \in (0,1)$ such that if*

$$\operatorname*{osc}_{Q^*_{2R}} w^* = \omega = \mu^+ - \mu^-,$$

then

$$\operatorname*{osc}_{Q^*_{\frac{R}{4}}(\theta_0,0,1-\frac{\rho}{2})} w^* \leqslant \omega - \lambda^* = \kappa\omega.$$

Proof. We first suppose that

$$\left|\left\{(x,t)\in Q_R(\theta_0);\ w(x,t)>\mu^+-\frac{\omega}{2}\right\}\right| < \varepsilon\left|Q_R(\theta_0)\right|$$

for sufficiently small $\varepsilon > 0$. Then, from Lemma 4.1, we obtain

$$w(x,t) < \mu^+ - \frac{\lambda\omega}{4} \quad \text{in } Q_{\frac{R}{2}}(\theta_0) \left(\supset Q_{\frac{R}{2}}\left(\theta_0, 0, 1 - \frac{\rho}{2}\right)\right).$$

Let's consider the function h_3 defined by

$$\begin{cases} \nabla\big(y^a \nabla h_3\big) = 0 & \text{in } B^*_{\frac{R}{2}}, \\ h_3 = \mu^+ & \text{on } \partial B^*_{\frac{R}{2}} \cap \{y > 0\}, \\ h_3 = \mu^+ - \dfrac{\lambda\omega}{4} & \text{on } y = 0. \end{cases}$$

Then, we have

$$w^* \leqslant h_3 \quad \text{in } Q^*_{\frac{R}{2}}\left(\theta_0, 0, 1 - \frac{\rho}{2}\right)$$

from the maximum principle. Since $h_3 \leqslant \mu^+ - \lambda_1$ in $B^*_{\frac{R}{4}}$ for some $0 < \lambda_1 < \frac{\lambda\omega}{4}$, we obtain that

$$\operatorname*{osc}_{Q^*_{\frac{R}{4}}(\theta_0, 0, 1-\frac{\rho}{2})} w^* \leqslant \mu^+ - \lambda_1 - \mu^- = \omega - \lambda_1.$$

Next, we assume that

$$\left| \left\{ (x,t) \in Q_R(\theta_0);\ w(x,t) > \mu^+ - \frac{\omega}{2} \right\} \right| > \varepsilon_1 \big| Q_R(\theta_0) \big|$$

for some $\varepsilon_1 > 0$. From Lemma 4.8, we have

$$w(x,t) \geqslant \mu^- + \frac{\lambda\omega}{2^{s^*+2}} \quad \text{in } Q_{\frac{R}{2}}\left(\theta_0, 0, \frac{\rho}{2}\right).$$

We also consider the function h_4 defined by

$$\begin{cases} \nabla\big(y^a \nabla h_4\big) = 0 & \text{in } B^*_{\frac{R}{2}}, \\ h_4 = \mu^- & \text{on } \partial B^*_{\frac{R}{2}} \cap \{y > 0\}, \\ h_4 = \mu^- + \dfrac{\lambda\omega}{2^{s^*+2}} & \text{on } y = 0. \end{cases}$$

Then, we have

$$w^* \geqslant h_4 \quad \text{in } Q^*_{\frac{R}{2}}\left(\theta_0, 0, 1 - \frac{\rho}{2}\right)$$

from the minimum principle. Since $h_4 \geqslant \mu^- + \lambda_2$ in $B^*_{\frac{R}{4}}$ for some $0 < \lambda_2 \leqslant \frac{\lambda\omega}{2^{s^*+2}}$, we have

$$\operatorname*{osc}_{Q^*_{\frac{R}{4}}(\theta_0,0,1-\frac{\rho}{2})} w^* \leqslant \mu^+ - \left(\mu^- + \lambda_1\right) = \omega - \lambda_1.$$

By taking $\lambda^* = \min\{\lambda_1, \lambda_2\}$, we get a desired conclusion

$$\operatorname*{osc}_{Q^*_{\frac{R}{4}}(\theta_0,0,1-\frac{\rho}{2})} w^* \leqslant \omega - \lambda^* = \kappa\omega. \qquad \square$$

Theorem 4.10 *(Hölder estimates). There exist constants $\gamma > 1$ and $\beta \in (0, 1)$ that can be determined a priori only in terms of the data, such that for all the cylinders*

$$\operatorname*{osc}_{Q^*_r(\theta_0,0,1-\frac{\rho}{2})} w^* \leqslant \gamma\omega\left(\frac{r}{R}\right)^{\beta} \quad (0 < r \leqslant R).$$

Proof. From the Oscillation Lemma (Lemma 4.9), we obtain

$$\operatorname*{osc}_{Q^*_{\frac{R}{2^k}}(\theta_0,0,1-\frac{\rho}{2})} w^* \leqslant \kappa^k\omega.$$

Let now $0 < r \leqslant R$ be fixed. There exists a non-negative integer k such that

$$\frac{R}{2^{k+1}} \leqslant r \leqslant \frac{R}{2^k}.$$

This implies the inequalities

$$-\log_2\left(\frac{r}{R}\right) \leqslant k+1 \quad \text{and} \quad \kappa^n \leqslant \frac{1}{\kappa}\kappa^{-\log_2(\frac{r}{R})} = \frac{1}{\kappa}\left(\frac{r}{R}\right)^{-\log_2\kappa}.$$

Since $\lambda^* \leqslant \frac{\lambda\omega}{2^{s^*+2}}$, we have $\frac{1}{2} < \kappa < 1$. Hence

$$\operatorname*{osc}_{Q^*_{\frac{R}{2^k}}(\theta_0,0,1-\frac{\rho}{2})} w^* \leqslant \gamma\omega\left(\frac{r}{R}\right)^{\beta}$$

where $\gamma = \frac{1}{\kappa} > 1$ and $0 < \beta = -\log_2\kappa < 1$. To conclude the proof we observe that the cylinder $Q^*_r(\theta_0, 0, 1 - \frac{\rho}{2})$ is included in $Q^*_{\frac{R}{2^k}}(\theta_0, 0, 1 - \frac{\rho}{2})$. $\square$

We finish with the proof of Theorem 1.2.

Proof of Theorem 1.2. From Theorem 4.10, the solution v^* of the problem (1.3) satisfies

$$\operatorname*{osc}_{Q_r^*(\theta_0,0,1-\frac{\rho}{2})} v^* = \operatorname*{osc}_{Q_r^*(\theta_0,0,1-\frac{\rho}{2})} w^* \leqslant \gamma\omega\left(\frac{r}{R}\right)^\beta \quad (0 < r \leqslant R)$$

since $v^* = M - w^*$. This gives that v^* is C^β at $(x, 0, t)$, and so v is C^β at (x, t). □

5. Asymptotic behavior for the FDE with fractional powers

5.1. Special solutions and stabilization

The asymptotic description is based on the existence of appropriate solutions that serve as model for the behavior near extinction: there is a self-similar solution of the form

$$U(x, t; T) = (T - t)^{1/(1-m)} f(x) \tag{5.1}$$

for a certain profile $f > 0$, where $\varphi = f^m$ is the solution of the super-linear elliptic equation

$$(-\Delta)^\sigma \varphi(x) = \frac{1}{1-m}\varphi(x)^p, \quad p = \frac{1}{m}$$

such that $\varphi > 0$ in Ω with zero on $\mathbb{R}^n \backslash \Omega$. Hence, similarity means in this case the separate-variables form. The existence and regularity of this solution depends on the exponent p, indeed it exists for $p < (n + 2\sigma)/(n - 2\sigma)$, the Sobolev exponent. Since $p = 1/m$, this means that smooth separate-variables solutions exist for

$$\frac{n-2\sigma}{n+2\sigma} < m < 1$$

an assumption that will be kept in the sequel. Note that the family of solutions (5.1) has a free parameter $T > 0$.

5.2. Stabilization

The above family of solutions allows to describe the behavior of general solutions near their extinction time.

Lemma 5.1. *Let u be a non-negative solution of* (1.1). *Suppose that the extinction time $T^* = 10$. Then*

$$\sup_\Omega v(x, 2) = \sup_\Omega u^m(x, 2) \geqslant c_0$$

where c_0 is a uniform constant.

Proof. Suppose that $\sup_{\Omega} v(x,2) < \epsilon$ for sufficiently small $\epsilon > 0$. Then, we have

$$\left(\int_{\mathbb{R}^n} v^{\frac{m+1}{m}}(x,2)\,dx\right)^{\frac{1-m}{1+m}} \leqslant \epsilon^{\frac{1-m}{m}} |\Omega|^{\frac{1-m}{1+m}}.$$

Hence, from the estimates on finite extinction time (Lemma 2.4), $T^* - 2 \leqslant C\epsilon^{\frac{1-m}{m}}|\Omega|^{\frac{1-m}{1+m}}$. If $\epsilon > 0$ is small enough, then we have $T^* \leqslant 5$, which is a contradiction to the assumption $T^* = 10$. □

Using the same arguments as in Section 5 in [12,13], we can obtain the following two corollaries.

Corollary 5.2. *Let u be a non-negative solution of* (1.1) *and u_0^m be a super-solution for the fractional Laplacian. Suppose that the extinction time $T^* = 10$. There exist a function $\psi(x) > 0$ in Ω and a constant $r \in (0,1)$ such that*

$$v(x,t) \geqslant C\psi(x) \quad \text{for } 2 - r \leqslant t \leqslant 2$$

where C is uniform.

Proof. Combining Lemma 5.1 with Theorem 4.10, we can take a point $x_0 \in \Omega$ and constants $\varepsilon, \sigma \in (0,1)$ and $\tilde{r} > 0$ such that

$$v(x,t) \geqslant c_1 > 0$$

for $(x,t) \in B_{\tilde{r}}(x_0) \times [2 - \varepsilon\tilde{r}^{2\sigma}, 2]$. Let ψ be the solution of

$$\begin{cases} (-\Delta)^\sigma \psi(x) = 0 & \text{in } \Omega \backslash \overline{B_{\tilde{r}}(x_0)}, \\ \psi = 1 & \text{on } B_{\tilde{r}}(x_0), \\ \psi = 0 & \text{on } \mathbb{R}^n \backslash \Omega. \end{cases}$$

On the other hand, v_t satisfies

$$\begin{cases} m v^{1-\frac{1}{m}} (-\Delta)^\sigma v_t + (v_t)_t = \dfrac{m-1}{mg} v_t^2 & \text{in } \Omega, \\ v_t = 0 & \text{on } \mathbb{R}^n \backslash \Omega. \end{cases}$$

Since the constant function $f = 0$ is a solution of above problem and $(-\Delta)^\sigma u_0^m \geqslant 0$, we have $(-\Delta)^\sigma v(x,t) \geqslant 0$ for all $0 < t < T^*$. Hence

$$c_1\psi(x,t) \leqslant v(x,t) \quad \text{in } \Omega \times \left[2 - \varepsilon\tilde{r}^{2\sigma}, 2\right].$$

Therefore, we have

$$C\psi(x) \leqslant v(x,t) \quad \text{in } \Omega \times [2 - r, 2]$$

where $C = c_1$ and $r = \varepsilon\tilde{r}^{2\sigma}$. □

Corollary 5.3. *Let u be a non-negative solution of* (1.1) *and u_0^m be a super-solution for the fractional Laplacian with the extinction time T^*. For $0 < t < T^*$*

$$u\left(x, \frac{T^* - t}{10}s + t\right) \geqslant C\psi(x)\left(T^* - t\right)^{\frac{1}{1-m}} \quad (2 - \delta \leqslant s \leqslant 2).$$

Proof. One can easily check that the scaled function $\tilde{g}$ defined by

$$\tilde{v}(x,t) = \left(\frac{T^* - t_0}{10}\right)^{\frac{-m}{1-m}} v\left(x, \left(\frac{T^* - t_0}{10}\right)t + t_0\right)$$

is a solution of (M.P) with the finite extinction time 10. Then, there exists a function $\psi(x) > 0$ such that

$$C\psi(x) \leqslant \tilde{v}(x,\tilde{t}) = \left(\frac{T^* - t_0}{10}\right)^{\frac{-m}{1-m}} v\left(x, \frac{T^* - t_0}{10}\tilde{t} + t_0\right) \quad (2 - r \leqslant \tilde{t} \leqslant 2)$$

from Corollary 5.2. Substituting in this inequality $v = u^m$, we easily come to a conclusion. $\square$

In the end of this section, we look at an asymptotic behavior of a sequence of time-slice for normalized solutions. We first put

$$u(x,t) = \left(T^* - t\right)^{\frac{1}{1-m}} \bar{u}(x,\tau) \quad \left(\tau = \ln\frac{T^*}{T^* - t}\right).$$

Then, the problem (1.1) is mapped into:

$$\begin{cases} \dfrac{\bar{u}}{1-m} = (-\Delta)^\sigma \bar{u}^m + \bar{u}_\tau & \text{in } \Omega, \\ \bar{u}(x,\tau) = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \\ \bar{u}(x,0) = \left(\dfrac{1}{T^*}\right)^{\frac{1}{1-m}} u_0(x) & \text{in } \Omega. \end{cases} \tag{5.2}$$

Observe that the new time τ ranges from 0 to ∞. We now state the main theorem in this section.

Theorem 5.4. *Under the above assumptions on u_0 and m, we have the following property near the extinction time of a solution $u(x,t)$: for any sequence $\{u(x,t_n)\}$, we have a subsequence $t_{n_k} \to T^*$ and a $\varphi(x)$ such that*

$$\lim_{k\to\infty} \left(T^* - t_{n_k}\right)^{-1/(1-m)} \left|u(x,t_{n_k}) - U\left(x,t_{n_k};T^*\right)\right| \to 0$$

uniformly in compact subset of Ω for $U(x,t;T^*) = (T^* - t)^{1/(1-m)}\varphi^{1/m}(x)$ where φ is an eigenfunction of fully non-linear equation

$$\begin{cases} (-\Delta)^\sigma \varphi = \dfrac{1}{1-m}\varphi^{\frac{1}{m}} & in\ \Omega, \\ \varphi = 0 & on\ \mathbb{R}^n\backslash\Omega, \\ \varphi > 0 & in\ \Omega. \end{cases}$$

Proof. The function $\bar{v} = \bar{u}^m$ satisfies the equation

$$\frac{\bar{v}^{\frac{1}{m}}}{1-m} = (-\Delta)^\sigma \bar{v} + \frac{1}{m}\bar{v}^{\frac{1-m}{m}}\bar{v}_\tau \tag{5.3}$$

in $\Omega \times (0, \infty)$, with $\bar{v} = 0$ on $\mathbb{R}^n\backslash\Omega$. Now define the functional

$$F(f) = \int\limits_{\mathbb{R}^n} \left(\frac{1}{2} f(-\Delta)^\sigma f - \frac{m}{(1-m)(1+m)} f^{\frac{1+m}{m}}\right) dx$$

and $g(\tau) = F(\bar{v}(\cdot, \tau))$. Then a simple calculation yields

$$g'(\tau) = -\frac{1}{m}\int\limits_{\mathbb{R}^n} \bar{v}^{\frac{1-m}{m}}(x, \tau)\bar{v}_\tau^2(x, \tau)\, dx,$$

the right side being non-positive since $\bar{v} \geqslant 0$. Lemma 2.5 shows that $\int_{\mathbb{R}^n} \bar{v}^{\frac{1+m}{m}}\, dx$ bounded in τ, so $g(\tau)$ is bounded below. Therefore $\lim_{\tau\to\infty} g(\tau)$ exists and there exists a sequence of times $\tau_n \to \infty$ such that $g'(\tau_n) \to 0$.

From Lemma 2.3, we now translate the estimate information in terms of $\bar{v}$ to the estimate

$$0 \leqslant \bar{v} \leqslant C$$

where $C > 0$ is a universal constant. In addition, for each τ, $\bar{v}(\cdot, \tau_n)$ is equicontinuous in every compact set $K \subset \Omega$ from Theorem 4.10. Hence, every subsequence, again labeled τ_n, $\{\bar{v}(\cdot, \tau_n)\}$ has a subsequence $\{\bar{v}(\cdot, \tau_{n_k})\}$ that converges to some function $\varphi(x)$ uniformly on every compact subset of K. Also its limit is non-trivial since $\bar{v}(x, \tau) \geqslant C\psi(x)$ when $\tau > \ln\frac{5}{4}$. Note also, by Lemma 2.5, that $\int_{\mathbb{R}^n} \bar{v}^{\frac{1+m}{m}}(x, \tau)\, dx$ is monotone decreasing, hence

$$\lim_{\tau\to\infty}\int\limits_{\mathbb{R}^n} \bar{v}^{\frac{1+m}{m}}(x, \tau)\, dx = \int\limits_{\mathbb{R}^n} \varphi^{\frac{1+m}{m}}(x)\, dx.$$

Multiply Eq. (5.3) by any test function $\eta \in C_c^\infty(K)$ and integrate in space, $x \in K$. Then, for each τ_{n_k},

$$\frac{1}{m}\int\limits_K \bar{v}^{\frac{1-m}{m}}\bar{v}_\tau \eta\, dx = \int\limits_K \bar{v}\big[-(-\Delta)^\sigma \eta\big]\, dx + \frac{1}{1-m}\int\limits_K \bar{v}^{\frac{1}{m}}\eta\, dx \tag{5.4}$$

from Theorem 1.2 in [11] (integration by part). Since the absolute value of the left-hand side of (5.4) is bounded above by

$$\left(\int_{\mathbb{R}^n} \eta^{\frac{2(1+m)}{m}}\,dx\right)^{\frac{m}{2(m+1)}} \left(\int_{\mathbb{R}^n} \bar{v}^{\frac{(1-m)(1+m)}{m}}\,dx\right)^{\frac{1}{2(m+1)}} \left(\int_{\mathbb{R}^n} \bar{v}^{\frac{1-m}{m}}\,\bar{v}_\tau^2\,dx\right)^{\frac{1}{2}} \tag{5.5}$$

and the third term of (5.5) has limit zero as $\tau_{n_k} \to \infty$, we get in the limit $\tau_{n_k} \to \infty$

$$\int_K \varphi(-\Delta)^\sigma \eta\,dx = \frac{1}{1-m}\int_K \varphi^{\frac{1}{m}} \eta\,dx,$$

which is weak formulation of the equation

$$(-\Delta)^\sigma \varphi = \frac{1}{1-m}\varphi^{\frac{1}{m}} \quad \text{in } K. \tag{5.6}$$

By the arbitrary choice of a compact subset K in Ω, (5.6) holds in Ω. □

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